The case of Description Logics

Fuzzy DLs

The semantics is an immediate consequence of the First-Order-Logic translation of DLs expressions

Interpretation:

$$\mathcal{I} = \Delta^{\mathcal{I}}$$
$$C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$$
$$R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$$

$\wedge$ = t-norm
$\vee$ = s-norm
$\neg$ = negation
$\rightarrow$ = implication

Concepts:

Syntax			Semantics		
C, D	$\longrightarrow$	$\top \mid$	$\top^{\mathcal{I}}(x)$	$=$	1
		$\bot \mid$	$\bot^{\mathcal{I}}(x)$	$=$	0
		$A \mid$	$A^{\mathcal{I}}(x)$	$\in$	$[0,1]$
		$C \sqcap D \mid$	$(C_1 \sqcap C_2)^{\mathcal{I}}(x)$	$=$	$C_1{}^{\mathcal{I}}(x) \wedge C_2{}^{\mathcal{I}}(x)$
		$C \sqcup D \mid$	$(C_1 \sqcup C_2)^{\mathcal{I}}(x)$	$=$	$C_1{}^{\mathcal{I}}(x) \vee C_2{}^{\mathcal{I}}(x)$
		$\neg C \mid$	$(\neg C)^{\mathcal{I}}(x)$	$=$	$\neg C^{\mathcal{I}}(x)$
		$\exists R.C \mid$	$(\exists R.C)^{\mathcal{I}}(x)$	$=$	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y)$
		$\forall R.C$	$(\forall R.C)^{\mathcal{I}}(x)$	$=$	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \rightarrow C^{\mathcal{I}}(y)\}$

Assertions: $\langle a{:}C, r\rangle$, $\mathcal{I} \models \langle a{:}C, r\rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq r$ (similarly for roles)

- individual a is instance of concept C at least to degree r, $r \in [0,1] \cap \mathbb{Q}$

Inclusion axioms: $\langle C \sqsubseteq D, r\rangle$,

- $\mathcal{I} \models \langle C \sqsubseteq D, r\rangle$ iff $\inf_{x \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \rightarrow D^{\mathcal{I}}(x) \geq r$

Main Inference Problems

Graded entailment: Check if DL axiom α is entailed to degree at least r

- $KB \models \langle \alpha, r \rangle$?

BED: Best Entailment Degree problem

- $bed(KB, \alpha) = \sup\{r \mid KB \models \langle \alpha, r \rangle\}$

BSD: Best Satisfiability Degree problem

- $bsd(KB, C) = \sup_{\mathcal{I} \models KB}\{C^{\mathcal{I}}(a^{\mathcal{I}})\}$, for new individual a

Top-k retrieval: Retrieve the top-k individuals that instantiate C w.r.t. best truth value bound

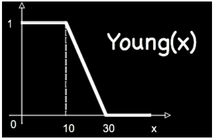
- $ans_k(KB, C) = Top_k\{\langle a, r \rangle \mid r = bed(KB, a{:}C)\}$

Towards fuzzy OWL Lite and OWL DL

- Recall that OWL Lite and OWL DL relate to $\mathcal{SHIF}(D)$ and $\mathcal{SHOIN}(D)$, respectively
- We need to extend the semantics of fuzzy $\mathcal{ALC}$ to fuzzy $\mathcal{SHOIN}(D) = \mathcal{ALCHOINR}_+(D)$
- Additionally, we add
  - modifiers (e.g., *very*)
  - concrete fuzzy concepts (e.g., *Young*)
  - both additions have explicit membership functions

Concrete fuzzy concepts

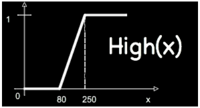
- E.g., *Small*, *Young*, *High*, *etc*. with explicit membership function
- Use the idea of concrete domains:
  - $D = \langle \Delta_\beta, \Phi_\beta \rangle$
  - Δ_β is an interpretation domain
  - Φ_β is the set of concrete fuzzy domain predicates d with a predefined arity $n = 1, 2$ and fixed interpretation $d^\beta : \Delta_\beta^n \to [0, 1]$
  - For instance,



$$\begin{array}{rcl} Minor & = & Person \sqcap \exists hasAge. \leq_{18} \\ YoungPerson & = & Person \sqcap \exists hasAge. Young \\ & & functional(hasAge) \end{array}$$

Modifiers

- *Very*, *moreOrLess*, *slightly*, etc.
- Apply to fuzzy sets to change their membership function
  - $very(x) = x^2$
  - $slightly(x) = \sqrt{x}$
- For instance,



$$SportsCar = Car \sqcap \exists speed.very(High)$$

Fuzzy $\mathcal{SHOIN}(D)$

Concepts:

Syntax			Semantics
C, D	$\longrightarrow$	$\top$ |	$\top(x)$
		$\bot$ |	$\bot(x)$
		A |	$A(x)$
		$(C \sqcap D)$ |	$C_1(x) \wedge C_2(x)$
		$(C \sqcup D)$ |	$C_1(x) \vee C_2(x)$
		$(\neg C)$ |	$\neg C(x)$
		$(\exists R.C)$ |	$\exists x\ R(x,y) \wedge C(y)$
		$(\forall R.C)$ |	$\forall x\ R(x,y) \rightarrow C(y)$
		$\{a\}$ |	$x = a$
		$(\geq n\ R)$ |	$\exists y_1, \ldots, y_n. \bigwedge_{i=1}^{n} R(x, y_i) \wedge \bigwedge_{1 \leq i < j \leq n} y_i \neq y_j$
		$(\leq n\ R)$ |	$\forall y_1, \ldots, y_{n+1}. \bigwedge_{i=1}^{n+1} R(x, y_i) \rightarrow \bigvee_{1 \leq i < j \leq n+1} y_i = y_j$
		FCC |	$\mu_{FCC}(x)$
		$M(C)$ |	$\mu_M(C(x))$
		$\sum_i w_i \cdot C_i$	$w_1 \cdot C_1(x) + \cdots w_n \cdot C_n(x) \quad (\sum_i w_i = 1)$
R	$\longrightarrow$	P |	$P(x,y)$
		P^- |	$P(y,x)$

Assertions:

Syntax			Semantics
α	$\longrightarrow$	$\langle a{:}C, r\rangle$ |	$r \rightarrow C(a)$
		$\langle (a,b){:}R, r\rangle$	$r \rightarrow R(a,b)$

Axioms:

Syntax			Semantics
τ	$\longrightarrow$	$\langle C \sqsubseteq D, r\rangle$ |	$\forall x\ r \rightarrow (C(x) \rightarrow D(x))$, where $\rightarrow$ is r-implication
		$fun(R)$ |	$\forall x \forall y \forall z\ R(x,y) \wedge R(x,z) \rightarrow y = z$
		$trans(R)$	$(\exists z\ R(x,z) \wedge R(z,y)) \rightarrow R(x,y)$

Example (Graded Entailment)

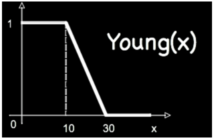
		
audi_tt	*mg*	*ferrari_enzo*

Car	*speed*
audi_tt	243
mg	≤ 170
ferrari_enzo	≥ 350

$$SportsCar \quad = \quad Car \sqcap \exists hasSpeed.very(High)$$

$$\begin{array}{lll} KB & \models & \langle ferrari\_enzo{:}SportsCar, 1\rangle \\ KB & \models & \langle audi\_tt{:}SportsCar, 0.92\rangle \\ KB & \models & \langle mg{:}\neg SportsCar, 0.72\rangle \end{array}$$

Example (Graded Subsumption)

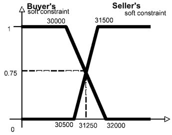


$$
\begin{aligned}
Minor &= Person \sqcap \exists hasAge. \leq_{18} \\
YoungPerson &= Person \sqcap \exists hasAge. Young
\end{aligned}
$$

$$KB \models \langle Minor \sqsubseteq YoungPerson, 0.6 \rangle$$

Note: without an explicit membership function of *Young*, this inference cannot be drawn

Example (Simplified Negotiation)



- a car seller sells an Audi TT for 31500 €, as from the catalog price.
- a buyer is looking for a sports-car, but wants to to pay not more than around 30000 €
- classical DLs: the problem relies on the crisp conditions on price

- more fine grained approach: to consider prices as fuzzy sets (as usual in negotiation)
  - seller may consider optimal to sell above 31500 €, but can go down to 30500 €
  - the buyer prefers to spend less than 30000 €, but can go up to 32000 €

    $$\begin{array}{lcl} \mathit{AudiTT} & = & \mathit{SportsCar} \sqcap \exists \mathit{hasPrice}.R(x; 30500, 31500) \\ \mathit{Query} & = & \mathit{SportsCar} \sqcap \exists \mathit{hasPrice}.L(x; 30000, 32000) \end{array}$$
  - highest degree to which the concept
    $C = \mathit{AudiTT} \sqcap \mathit{Query}$
    is satisfiable is 0.75 (the possibility that the Audi TT and the query matches is 0.75)

  - the car may be sold at 31250 €

Reasoning in Fuzzy $\mathcal{ALC}$, under Zadeh Semantics

- Applies technique based on Mixed Integer Programming (MILP) for fuzzy propositional logic to $\mathcal{ALC}$ calculus
- For each concept assertion α of the form $a{:}C$, we use variable x_α, which holds the degree of truth of α
- It can be shown that

$$
\begin{array}{lcl}
bed(KB, (a,b){:}R) & = & bed(KB \cup \{\langle b{:}B, 1\rangle\}, a{:}\exists R.B) \\
bed(KB, C \sqsubseteq D) & = & \min x \text{ such that } KB \cup \{\langle b{:}C \sqcap \neg D, 1-x\rangle\} \text{ satisfiable} \\
bed(KB, a{:}C) & = & \min x \text{ such that } KB \cup \{\langle a{:}\neg C, 1-x\rangle\} \text{ satisfiable} \\
bsd(KB, C) & = & \min -x \text{ such that } KB \cup \{\langle b{:}C, x\rangle\} \text{ satisfiable}
\end{array}
$$

where b is a new individual and B is a new concept

Satisfiability Testing

- The notion of completion forest $\mathcal{F}$ is similar to the case of $\mathcal{ALC}$
  - $\mathcal{F}$ contains a root node a_i for each individual a_i occurring in $\mathcal{A}$
  - $\mathcal{F}$ contains an edge $\langle a, b \rangle$ for each $\langle (a, b){:}R, n \rangle \in \mathcal{A}$
  - for each $\langle a{:}C, n \rangle \in \mathcal{A}$, we add both C to $\mathcal{L}(a)$ and $x_{a:C} \geq n$ to $\mathcal{C}_{\mathcal{F}}$
  - for each $\langle (a, b){:}R, n \rangle \in \mathcal{A}$, we add both R to $\mathcal{L}(\langle a, b \rangle)$ and $x_{(a,b):R} \geq n$ to $\mathcal{C}_{\mathcal{F}}$
- The notion of blocking is as for crisp $\mathcal{ALC}$
- $\mathcal{F}$ is then expanded by repeatedly applying the rules described below
- The completion-forest is complete when none of the rules are applicable
- Then, the bMILP problem on $\mathcal{C}_{\mathcal{F}}$ is solved

Fuzzy $\mathcal{ALC}$ Tableau rules with GCI's (Zadeh semantics)

Rule		Description
(var)		For variable $x_{v:C}$ add $x_{v:C} \in [0,1]$ to $\mathcal{C}_{\mathcal{F}}$. For variable $x_{(v,w):R}$, add $x_{(v,w):R} \in [0,1]$ to $\mathcal{C}_{\mathcal{F}}$
$(\bar{A})$	if	$\neg A \in \mathcal{L}(v)$ then add $x_{v:A} = 1 - x_{v:\neg A}$ to $\mathcal{C}_{\mathcal{F}}$
$(\bot)$	If	$\bot \in \mathcal{L}(v)$ then add $x_{v:\bot} = 0$ to $\mathcal{C}_{\mathcal{F}}$
$(\top)$	If	$\top \in \mathcal{L}(v)$ then add $x_{v:\top} = 1$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqcap)$	if	$C_1 \sqcap C_2 \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(v) \rightarrow \mathcal{L}(v) \cup \{C_1, C_2\}$, and add $x_{v:C_1} \otimes x_{v:C_2} \geq x_{v:C_1 \sqcap C_2}$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqcup)$	if	$C_1 \sqcup C_2 \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(v) \rightarrow \mathcal{L}(v) \cup \{C_1, C_2\}$, and add $x_{v:C_1} \oplus x_{v:C_2} \geq x_{v:C_1 \sqcup C_2}$ to $\mathcal{C}_{\mathcal{F}}$
$(\forall)$	if	$\forall R.C \in \mathcal{L}(v)$, v is not indirectly blocked
	then	$\mathcal{L}(w) \rightarrow \mathcal{L}(w) \cup \{C\}$, and add $x_{w:C} \geq x_{v:\forall R.C} \otimes x_{(v,w):R}$ to $\mathcal{C}_{\mathcal{F}}$
$(\exists)$	if	$\exists R.C \in \mathcal{L}(v)$, v is not blocked
	then	create new node w with $\mathcal{L}(\langle v, w\rangle) = \{R\}$ and $\mathcal{L}(w) = \{C\}$, and add $x_{w:C} \otimes x_{(v,w):R} \geq x_{v:\exists R.C}$ to $\mathcal{C}_{\mathcal{F}}$
$(\sqsubseteq)$	if	$\langle C \sqsubseteq D, n\rangle \in \mathcal{T}$, v is not indirectly blocked
	then	$\mathcal{L}(v) \rightarrow \mathcal{L}(v) \cup \{C, D\}$, and add $x_{v:D} \geq x_{v:C} \otimes n$ to $\mathcal{C}_{\mathcal{F}}$

Fuzzy Concrete Domains

- As fuzzy DL extensions involving
  - concrete domains
  - modifiers

  are combinations of linear functions, they can be translated into MILP equations
- Therefore, the algorithm can be extended to fuzzy DLs with concrete domains
- For a fuzzy DL system, see `http://www.straccia.info/software/fuzzyDL/fuzzyDL.html`

Example: Multi-Criteria Decision Making

- The problem is about selecting an appropriate location
  - We have to select among two sites, A_1, A_2
  - according to two criteria (C_1 -Transportation Issues, and C_2 -Public Nuisance)
  - and there are two experts (E_1, E_2)
- The decision matrix of the experts is shown below:

E_1		Criteria	
		0.48	0.52
Alter.		C_1	C_2
x_1	A_1	$tri(0.6, 0.7, 0.8)$	$tri(0.9, 0.95, 1.0)$
x_2	A_2	$tri(0.6, 0.7, 0.8)$	$tri(0.4, 0.5, 0.6)$

E_2		Criteria	
		0.52	0.48
Alter.		C_1	C_2
x_1	A_1	$tri(0.55, 0.6, 0.7)$	$tri(0.4, 0.45, 0.5)$
x_2	A_2	$tri(0.35, 0.4, 0.45)$	$tri(0.5, 0.55, 0.6)$

- For each expert $k = 1, 2$, for each alternative $i = 1, 2$ and for each criteria $j = 1, 2$, we define the concept

$$P_{ij}^k = \exists \text{hasScore}.a_{ij}^k$$

- Now, for each expert k and alternative i, we define the weighted concept

$$A_i^k = w_1^k \cdot P_{i1}^k + w_2^k \cdot P_{i2}^k$$

- Finally, we combine the two experts outcome, by defining the weighted concept

$$A_i = 0.5 \cdot A_i^1 + 0.5 \cdot A_i^2$$

- It can be verified that $rv(KB, A_1) = bsd(KB, A_1) = 0.26$ and $rv(KB, A_2) = bsd(KB, A_2) = 0.37$

Exercise

- Encode
  - Matchmaking
  - Multi-Criteria Decision Making

  examples of fuzzy propositional logic into fuzzy DL