# A Fuzzy Description Logic for the Semantic Web

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**Abstract**

In this paper we present a fuzzy version of $\mathcal{SHOIN}(\mathtt{D})$, the corresponding Description Logic of the ontology description language OWL DL. We show that the representation and reasoning capabilities of fuzzy $\mathcal{SHOIN}(\mathtt{D})$ go clearly beyond classical $\mathcal{SHOIN}(\mathtt{D})$. Interesting features are: (*i*) concept constructors are based on t-norm, t-conorm, negation and implication; (*ii*) concrete domains are fuzzy sets; (*iii*) fuzzy modifiers are allowed; and (*iv*) entailment and subsumption relationships may hold to some degree in the unit interval $[0, 1]$.

**Keywords**: description logics, ontoloies, fuzzy logics

## 1 Introduction

In the last decade a substantial amount of work has been carried out in the context of *Description Logics* (DLs) [1]. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge.

Nowadays, DLs have gained even more popularity due to their application in the context of the *Semantic Web* [3, 16]. Semantic Web has recently attracted much attention both from academia and industry, and is widely regarded as the next step in the evolution of the World Wide Web. It aims at enhancing content on the World Wide Web with meta-data, enabling *agents* (machines or human users) to *process*, *share* and *interpret* Web content.

*Ontologies* [9] play a key role in the Semantic Web and major effort has been put by the Semantic Web community into this issue. Informally, an ontology consists of a hierarchical description of important concepts in a particular domain, along with the description of the properties (of the instances) of each concept. DLs play a particular role in this context as they are essentially the theoretical counterpart of the *Web Ontology Language OWL DL*, the state of the art language to specify ontologies. Web content is then annotated by relying on the concepts defined in a specific domain ontology.

However, OWL DL becomes less suitable in all those domains in which the concepts to be represented have not a precise definition. If we take into account that we have to deal with Web content, then it is easily verified that this scenario is, unfortunately, likely the rule rather than an exception. For instance, just consider the case we would like to build an ontology about flowers. Then we may encounter the problem of representing concepts like [1] "Candia is a creamy white rose with dark pink edges to the petals", "Jacaranda is a hot pink rose", "Calla is a very large, long white flower on thick stalks". As it becomes apparent such concepts hardly can be encoded into OWL DL, as they involve so-called *fuzzy* or *vague concepts*, like "creamy", "dark", "hot", "large" and "thick", for which a clear and precise definition is not possible (another issue relates to the representation of terms like "very", which are called fuzzy concepts *modifiers*, as we will see later on).

The problem to deal with imprecise concepts has been addressed several decades ago by Zadeh [34], which gave birth in the meanwhile to the so-called *fuzzy set and fuzzy logic theory* and a huge number of real life applications exists. Unfortunately, despite the popularity of fuzzy set theory, relative little work has been carried out in extending DLs towards the representation of imprecise concepts, notwithstanding DLs can be considered as a quite natural candidate for such an extension [4, 12, 13, 25, 27, 29, 30, 32, 33] (see also [7], Chapter 6).

In this paper we consider a fuzzy extension of $\mathcal{SHOIN}(\mathtt{D})$, the corresponding DL of the ontology description language OWL DL, and present its syntax and semantics. The main feature of fuzzy $\mathcal{SHOIN}(\mathtt{D})$ is that it allows us to represent and reason about vague concepts. None of the approaches to fuzzy DLs deal with the expressive power of the fuzzy extension of $\mathcal{SHOIN}(\mathtt{D})$ we present here. Our purpose is also to integrate most of these contributions within an unique setting and, thus, hope to define a reference language for fuzzy $\mathcal{SHOIN}(\mathtt{D})$. Interesting features are: (*i*) concept constructors are based on t-norm, t-conorm, negation and implication; (*ii*) concrete domains are fuzzy sets; (*iii*) fuzzy modifiers are allowed; and (*iv*) entailment and subsumption relationships may hold to some degree in the unit interval $[0, 1]$.

We will proceed as follows. In the following section we recall the description logic $\mathcal{SHOIN}(\mathtt{D})$. In Section 3 we extend $\mathcal{SHOIN}(\mathtt{D})$ to the fuzzy case and discuss some properties of it. Section 4 presents related work, while Section 5 concludes and presents some topics for further research.

## 2 Preliminaries

The ontology language OWL DL is strictly related to the DL $\mathcal{SHOIN}(\mathtt{D})$ [16]. Although several XML and RDF syntaxes for OWL-DL exist, in this paper we use the traditional description logic notation. For explicating the relationship between OWL DL and DLs syntax, see e.g. [15, 16]. The purpose of this section is to make the paper self-contained. More importantly it helps in understanding

[1] Taken from a text book on flowers.

the differences between classical $\mathcal{SHOIN}(\mathtt{D})$ and fuzzy $\mathcal{SHOIN}(\mathtt{D})$. The reader confident with the $\mathcal{SHOIN}(\mathtt{D})$ terminology may skip directly to Section 3.

## 2.1 Syntax

$\mathcal{SHOIN}(\mathtt{D})$ allows to reason with concrete data types, such as strings and integers using so-called *concrete domains* [2, 21, 22, 23].

**Concrete domains.** A *concrete domain* $\mathtt{D}$ is a pair $\langle \Delta_{\mathtt{D}}, \Phi_{\mathtt{D}} \rangle$, where $\Delta_{\mathtt{D}}$ is an interpretation domain and $\Phi_{\mathtt{D}}$ is the set of concrete domain predicates $d$ with a predefined arity $n$ and an interpretation $d^{\mathtt{D}} \subseteq \Delta_{\mathtt{D}}^n$. For instance, over the integers $\geq_{20}$ may be an unary predicate denoting the set of integers greater or equal to 20. For instance, $\mathtt{Person} \sqcap \exists \mathtt{age}. \geq_{20}$ will denote a person whose age is greater or equal to 20.

**Alphabeths.** Let $\mathtt{C}$, $\mathtt{R}_a$, $\mathtt{R}_c$, $\mathtt{I}_a$ and $\mathtt{I}_c$ be non-empty finite and pair-wise disjoint sets of *concepts names*, *abstract roles names*, *concrete roles names*, *abstract individual names* and *concrete individual names*.

**RBox.** An *abstract role* is an abstract role name or the inverse $S^-$ of an abstract role name $S$ (concrete role names do not have inverses). An *RBox* $\mathcal{R}$ consists of a finite set of transitivity axioms $\mathtt{trans}(R)$, and role inclusion axioms of the form $R \sqsubseteq S$ and $T \sqsubseteq U$, where $R$ and $S$ are abstract roles, and $T$ and $U$ are concrete roles. The reflexive-transitive closure of the role inclusion relationship is denoted with $\sqsubseteq^*$. A role not having transitive sub-roles is called *simple role*.

**Concepts.** The set of $\mathcal{SHOIN}(\mathtt{D})$ *concepts* is defined by the following syntactic rules, where $A$ is an atomic concept, $R$ is an abstract role, $S$ is an abstract simple role, $T_i$ are concrete roles, $d$ is a concrete domain predicate, $a_i$ and $c_i$ are abstract and concrete individuals, respectively, and $n \in \mathbb{N}$:

$$
\begin{array}{rcl}
C & \longrightarrow & \top \mid \bot \mid A \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \neg C \mid \\
 & & \forall R.C \mid \exists R.C \mid (\geq n\ S) \mid (\leq n\ S) \mid \{a_1, \ldots, a_n\} \mid \\
 & & (\geq n\ T) \mid (\leq n\ T) \mid \forall T_1, \ldots, T_n.D \mid \exists T_1, \ldots, T_n.D \\
 & & \\
D & \longrightarrow & d \mid \{c_1, ..., c_n\}
\end{array}
$$

For instance, we may write the concept

$$\mathtt{Flower} \sqcap (\exists \mathtt{hasPetalWidth}.(\geq_{\mathtt{20mm}} \sqcap \leq_{\mathtt{40mm}})) \sqcap \exists \mathtt{hasColour}.\mathtt{Red})$$

to informally denote the set of flowers having petal's dimension within 20mm and 40mm, whose colour is red. Here $\geq_{\mathtt{20mm}}$ (and $\leq_{\mathtt{40mm}}$) is a concrete domain predicate. We use $(= 1\ S)$ as an abbreviation for $(\geq 1\ S) \sqcap (\leq 1\ S)$.

**TBox.** A *TBox* $\mathcal{T}$ consists of a finite set of concept inclusion axioms $C \sqsubseteq D$, where $C$ and $D$ are concepts. For ease, we use $C = D \in \mathcal{T}$ in place of $C \sqsubseteq D, D \sqsubseteq C \in \mathcal{T}$.

An abstract simple role $S$ is called *functional* if the interpretation of role $S$ is always functional (see later for the semantics). A functional role $S$ can always be obtained from an abstract role by means of the axiom $\top \sqsubseteq (\leq 1\ S)$. Therefore, whenever we say that a role is functional, we assume that $\top \sqsubseteq (\leq 1\ S)$ is in the TBox.

**ABox.** An *ABox* $\mathcal{A}$ consists of a finite set of concept and role assertion axioms and individual (in)equality axioms $a{:}C, (a,b){:}R, (a,c){:}T, a \approx b$ and $a \not\approx b$, respectively.

**Knowledge base.** A $\mathcal{SHOIN}(\mathtt{D})$ *knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of a TBox $\mathcal{T}$, an RBox $\mathcal{R}$, and an ABox $\mathcal{A}$.

## 2.2 Semantics

**Interpretation.** An *interpretation* $\mathcal{I}$ with respect to a concrete domain $\mathtt{D}$ is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of a non empty set $\Delta^\mathcal{I}$ (called the *domain*), disjoint from $\Delta_\mathtt{D}$, and of an *interpretation function* $\cdot^\mathcal{I}$ that assigns to each $C \in \mathtt{C}$ a subset of $\Delta^\mathcal{I}$, to each $R \in \mathtt{R}_a$ a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$, to each $a \in \mathtt{I}_a$ an element in $\Delta^\mathcal{I}$, to each $c \in \mathtt{I}_c$ an element in $\Delta_\mathtt{D}$, to each $T \in \mathtt{R}_c$ a subset of $\Delta^\mathcal{I} \times \Delta_\mathtt{D}$ and to each $n$-ary concrete predicate $d$ the interpretation $d^\mathtt{D} \subseteq \Delta^n_\mathtt{D}$.

The mapping $\cdot^\mathcal{I}$ is extended to concepts and roles as usual:

$$\begin{array}{rcl}
\top^\mathcal{I} & = & \Delta^\mathcal{I} \\
\bot^\mathcal{I} & = & \emptyset \\
(C_1 \sqcap C_2)^\mathcal{I} & = & {C_1}^\mathcal{I} \cap {C_2}^\mathcal{I} \\
(C_1 \sqcup C_2)^\mathcal{I} & = & {C_1}^\mathcal{I} \cup {C_2}^\mathcal{I} \\
(\neg C)^\mathcal{I} & = & \Delta^\mathcal{I} \setminus C^\mathcal{I} \\
(S^-)^\mathcal{I} & = & \{\langle y,x \rangle \colon \langle x,y \rangle \in S^\mathcal{I}\} \\
(\forall R.C)^\mathcal{I} & = & \{x \in \Delta^\mathcal{I} : R^\mathcal{I}(x) \subseteq C^\mathcal{I}\} \\
(\exists R.C)^\mathcal{I} & = & \{x \in \Delta^\mathcal{I} : R^\mathcal{I}(x) \cap C^\mathcal{I} \neq \emptyset\} \\
(\geq n\ S)^\mathcal{I} & = & \{x \in \Delta^\mathcal{I} : |S^\mathcal{I}(x)| \geq n\} \\
(\leq n\ S)^\mathcal{I} & = & \{x \in \Delta^\mathcal{I} : |S^\mathcal{I}(x)| \leq n\} \\
\{a_1, \ldots, a_n\}^\mathcal{I} & = & \{{a_1}^\mathcal{I}, \ldots, {a_n}^\mathcal{I}\}
\end{array}$$

and similarly for the other constructs, where $R^\mathcal{I}(x) = \{y \colon \langle x,y \rangle \in R^\mathcal{I}\}$ and $|X|$ denotes the cardinality of the set $X$. In particular,

$$(\exists T_1, \ldots, T_n.d)^\mathcal{I} = \{x \in \Delta^\mathcal{I} : [{T_1}^\mathcal{I}(x) \times \ldots \times {T_n}^\mathcal{I}(x)] \cap d^\mathtt{D} \neq \emptyset\} \ .$$

**Satisfiability.** The *satisfiability* of an axiom $E$ in an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, denoted $I \models E$, is defined as follows: $I \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, $I \models R \sqsubseteq S$ iff $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$, $I \models T \sqsubseteq U$ iff $T^{\mathcal{I}} \subseteq U^{\mathcal{I}}$, $I \models \mathtt{trans}(R)$ iff $R^{\mathcal{I}}$ is transitive, $I \models a{:}C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$, $I \models (a,b){:}R$ iff $\langle a^{\mathcal{I}}, b^{\mathcal{I}} \rangle \in R^{\mathcal{I}}$, $I \models (a,c){:}T$ iff $\langle a^{\mathcal{I}}, c^{\mathcal{I}} \rangle \in T^{\mathcal{I}}$, $I \models a \approx b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$, $I \models a \not\approx b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

For a set of axioms $\mathcal{E}$, we say that $I$ *satisfies* $\mathcal{E}$ iff $I$ satisfies each element in $\mathcal{E}$. If $I \models E$ (resp. $I \models \mathcal{E}$) we say that $I$ is a *model* of $E$ (resp. $\mathcal{E}$). $I$ *satisfies* (is a *model* of) a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, denoted $I \models \mathcal{K}$, iff $I$ is a model of each component $\mathcal{T}, \mathcal{R}$ and $\mathcal{A}$, respectively.

**Logical consequence.** An axiom $E$ is a *logical consequence* of a knowledge base $\mathcal{K}$, denoted $\mathcal{K} \models E$, iff every model of $\mathcal{K}$ satisfies $E$. According to [15], the entailment and subsumption problem can be reduced to knowledge base satisfiability problem (e.g. $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle \models a{:}C$ iff $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{a{:}\neg C\} \rangle$ unsatisfiable), for which decision procedures and reasoning tools exists (e.g. RACER [10] and FACT [14]).

**Example 1** *Let us consider the following excerpt of a simple ontology (TBox $\mathcal{T}$) about cars, with empty RBox ($\mathcal{R} = \emptyset$):*

$$\mathtt{Car} \sqsubseteq (=1\ \mathtt{maker}) \sqcap (=1\ \mathtt{passenger}) \sqcap (=1\ \mathtt{speed})$$

$$\begin{array}{ll} (=1\ \mathtt{maker}) \sqsubseteq \mathtt{Car} & \top \sqsubseteq \forall \mathtt{maker}.\mathtt{Maker} \\ (=1\ \mathtt{passenger}) \sqsubseteq \mathtt{Car} & \top \sqsubseteq \forall \mathtt{passenger}.\mathbb{N} \\ (=1\ \mathtt{speed}) \sqsubseteq \mathtt{Car} & \top \sqsubseteq \forall \mathtt{speed}.\mathtt{Km/h} \end{array}$$

$$\begin{array}{c} \mathtt{Roadster} \sqsubseteq \mathtt{Cabriolet} \sqcap \exists \mathtt{passenger}.\{2\} \\ \mathtt{Cabriolet} \sqsubseteq \mathtt{Car} \sqcap \exists \mathtt{topType}.\mathtt{SoftTop} \\ \mathtt{SportsCar} = \mathtt{Car} \sqcap \exists \mathtt{speed}.\geq_{\mathtt{245km/h}} \end{array}$$

*In $\mathcal{T}$, the value for* `speed` *ranges over the concrete domain of kilometres per hour,* `Km/h`*, while the value for* `passengers` *ranges over the concrete domain of natural numbers, $\mathbb{N}$. The concrete predicate $\geq_{\mathtt{245km/h}}$ is true if the value is greater or equal than to* `245km/h`*.*

*The ABox $\mathcal{A}$ contains the following assertions:*

$$\begin{array}{l} \mathtt{mgb}{:}\mathtt{Roadster} \sqcap (\exists \mathtt{maker}.\{\mathtt{mg}\}) \sqcap (\exists \mathtt{speed}.\leq_{\mathtt{170km/h}}) \\ \mathtt{enzo}{:}\mathtt{Car} \sqcap (\exists \mathtt{maker}.\{\mathtt{ferrari}\}) \sqcap (\exists \mathtt{speed}.>_{\mathtt{350km/h}}) \\ \mathtt{tt}{:}\mathtt{Car} \sqcap (\exists \mathtt{maker}.\{\mathtt{audi}\}) \sqcap (\exists \mathtt{speed}.=_{\mathtt{243km/h}}) \end{array}$$

*Consider the knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$. It is then easily verified that, e.g.*

$$\begin{array}{ll} \mathcal{K} \models \mathtt{Roadster} \sqsubseteq \mathtt{Car} & \mathcal{K} \models \mathtt{mg}{:}\mathtt{Maker} \\ \mathcal{K} \models \mathtt{enzo}{:}\mathtt{SportsCar} & \mathcal{K} \models \mathtt{tt}{:}\neg\mathtt{SportsCar}\ . \end{array}$$

□

The above example illustrates an evident difficulty in defining the class of sport cars. Indeed, it is highly questionable why a car whose speed is `243km/h` is *not* a sport car any more. The point is that essentially, the higher the speed the more likely a car is a sports car, which makes the concept of sports car rather a *fuzzy* concept, i.e. *vague* concept, rather than a crisp one. In the next section we will see how to represent such concepts more appropriately.

# 3 Fuzzy OWL DL

Fuzzy sets have been introduced by Zadeh [34] as a way to deal with vague concepts like `low pressure`, `high speed` and the like. Formally, a *fuzzy set* $A$ with respect to a universe $X$ is characterized by a *membership function* $\mu_A : X \to [0,1]$, assigning an $A$-membership degree, $\mu_A(x)$, to each element $x$ in $X$. $\mu_A(x)$ gives us an estimation of the belonging of $x$ to $A$. Typically, if $\mu_A(x) = 1$ then $x$ definitely belongs to $A$, while $\mu_A(x) = 0.8$ means that $x$ is "likely" to be an element of $A$.

When we switch to fuzzy logics, the notion of degree of membership $\mu_A(x)$ of an element $x \in X$ w.r.t. the fuzzy set $A$ over $X$ is regarded as the *degree of truth* in $[0,1]$ of the statement *"x is A"*. Accordingly, in our fuzzy DL, ($i$) a concept $C$, rather than being interpreted as a classical set, will be interpreted as a fuzzy set and, thus, concepts become *imprecise*; and, consequently, ($ii$) the statement "$a$ is $C$", i.e. $a{:}C$, will have a truth-value in $[0,1]$ given by the degree of membership of being the individual $a$ a member of the fuzzy set $C$.

In the following, we present first some preliminaries on fuzzy set theory (for a more complete and comprehensive presentation see e.g. [6, 11, 17, 8]) and then define fuzzy $\mathcal{SHOIN}(\mathtt{D})$.

## 3.1 Preliminaries on fuzzy set theory

Let $X$ be a crisp set and let $A$ be a fuzzy subset of $X$, with membership function $\mu_A(x)$, or simply $A(x) \in [0,1], x \in X$. The *support* of $A$, $\mathtt{supp}(A)$, is the crisp set $\mathtt{supp}(A) = \{x \in X : A(x) \neq 0\}$. The *scalar cardinality* of $A$, $|A|$, is defined as $|A| = \sum_{x \in X} A(x)$. The *fuzzy powerset* of $X$, $\mathcal{F}(X)$, is the set of all the fuzzy sets over $X$. Let $A, B \in \mathcal{F}(X)$. We say that $A$ and $B$ are *equivalent* iff $A(x) = B(x), \forall x \in X$. $A$ is a *crisp subset* of $B$ iff $A(x) \leq B(x), \forall x \in X$. Note that either $A$ is a subset of $B$ or it is not. We will see later on a different notion of subset, in which $A$ is a subset of $B$ to some degree in $[0,1]$. We next give the basic definitions on fuzzy set operations (complement, intersection and union).

The *complement* of $A$, $\neg A$, is given by membership function $(\neg A)(x) = n(A(x))$, for any $x \in X$. The function $n: [0,1] \to [0,1]$, called *negation*, has to satisfy the following conditions and extends boolean negation:

- $n(0) = 1$ and $n(1) = 0$;
- $\forall a, b \in [0,1], a \leq b$ implies $n(b) \leq n(a)$;
- $\forall a \in [0,1]$, $n(n(a)) = a$.

Several negation functions have been given in the literature, e.g. Lukasiewicz negation $n_L(a) = 1-a$ (syntax, $\neg_L$) and Gödel negation $n_G(0) = 1$ and $n(a) = 0$ if $a > 0$ (syntax, $\neg_G$).

The *intersection* of two fuzzy sets $A$ and $B$ is given $(A \wedge B)(x) = t(A(x), B(x))$, where $t$ is a *triangular norm*, or simply *t-norm*. A t-norm is, called *conjunction*, is a function $t\colon [0,1] \times [0,1] \to [0,1]$ that has to satisfy the following conditions:

- $\forall a \in [0,1], t(a,1) = a$;
- $\forall a,b,c \in [0,1], b \leq c$ implies $t(a,b) \leq t(a,c)$;
- $\forall a,b \in [0,1], t(a,b) = t(b,a)$;
- $\forall a,b,c \in [0,1], t(a,t(b,c)) = t(t(a,b),c)$.

Examples of t-norms are: $t_L(a,b) = \max(a+b-1,0)$ (Lukasiewicz t-norm, syntax $\wedge_L$), $t_G(a,b) = \min(a,b)$ (Gödel t-norm, syntax $\wedge_G$), and $t_P(a,b) = a \cdot b$ (product t-norm, syntax $\wedge_P$). Note that $\forall a \in [0,1], t(a,0) = 0$.

The *union* of two fuzzy sets $A$ and $B$ is given $(A \vee B)(x) = s(A(x), B(x))$, where $s$ is a *triangular co-norm*, or simply *s-norm*. A s-norm, called *disjunction*, is a function $s\colon [0,1] \times [0,1] \to [0,1]$ that has to satisfy the following conditions:

- $\forall a \in [0,1], s(a,0) = a$;
- $\forall a,b,c \in [0,1], b \leq c$ implies $s(a,b) \leq s(a,c)$;
- $\forall a,b \in [0,1], s(a,b) = s(b,a)$;
- $\forall a,b,c \in [0,1], s(a,s(b,c)) = s(s(a,b),c)$.

Examples of s-norms are: $s_L(a,b) = \min(a+b,1)$ (Lukasiewicz s-norm, syntax $\vee_L$), $s_G(a,b) = \max(a,b)$ (Gödel s-norm, syntax $\vee_G$), and $s_P(a,b) = a+b-a \cdot b$ (product s-norm, syntax $\vee_P$). Note that if we consider Lukasiewicz negation, then Lukasiewicz, Gödel and product s-norm are related to their respective t-norm according to the De Morgan law: $\forall a,b \in [0,1], s(a,b) = n(t(n(a),n(b)))$.

Another important operator is *implication*, denoted $\to$, that gives a truth-value to the formula $A \to B$, when the truth of $A$ and $B$ are known. A fuzzy implication is a function $i\colon [0,1] \times [0,1] \to [0,1]$ that has to satisfy the following conditions:

- $\forall a,b,c \in [0,1], a \leq b$ implies $i(a,c) \geq i(b,c)$;
- $\forall a,b,c \in [0,1], b \leq c$ implies $i(a,b) \leq i(a,c)$;
- $\forall a \in [0,1], i(0,b) = 1$;
- $\forall a \in [0,1], i(a,1) = 1$;
- $i(1,0) = 0$.

In classical logic, $a \rightarrow b$ is a shorthand for $\neg a \vee b$. A generalization to fuzzy logic is, thus, $\forall a, b \in [0,1], i(a,b) = s(n(a), b)$. For instance, $\forall a, b \in [0,1], i_{KD}(a,b,) = \max(1-a, b)$ is the so-called Kleene-Dienes implication (syntax, $\rightarrow_{KD}$). Another approach to fuzzy implication is based on the so-called *residuum*. His formulation starts from the fact that in classical logic $\neg a \vee b$ can be re-written as $\max\{c \in \{0,1\} \colon a \wedge c \leq b\}$. Therefore, another generalization of implication to fuzzy logic is

$$\forall a, b \in [0,1], i(a,b) = \sup\{c \in [0,1] \colon t(a,c) \leq b\} \ .$$

For residuum based implication, $i(a,b) = 1$ if $a \leq b$. If $a > b$ then, according to the chosen t-norm, we have that e.g. $i_L(a,b) = 1 - a + b$ for Lukasiewicz implication (syntax, $\rightarrow_L$), $i_G(a,b) = b$ for Gödel implication (syntax, $\rightarrow_G$)) and $i_P(a,b) = a/b$ for product implication (syntax, $\rightarrow_P$). Note that, for Lukasiewcz implication, s-norm and negation, we have $i_L(a,b) = s_L(n_L(a), b)$. The same holds using Kleene-Dienes implication, Lukasiewicz negation and Gödel s-norm. On the other hand $i_P(a,b) \neq s_P(n_G(a), b)$ (for instance, for $0 < a \leq b < 1$, $i_P(a,b) = 1$, while $s_P(n_G(a), b) = b < 1$).

Another interesting question is when $\forall a, b \in [0,1], i(a,b) = n(t(a, n(b))$ holds, which in formulae is formulated as $a \rightarrow b \equiv \neg(a \wedge \neg b)$. It turns out that e.g., in Zadeh's logic [34] (i.e. using $\rightarrow_{KD}, \wedge_G, \neg_L$) this relation holds. It holds as well in the so-called Lukasiewcz logic (i.e. using $\rightarrow_L, \wedge_L, \neg_L$), while it does neither hold for Gödel logic (i.e. using $\rightarrow_G, \wedge_G, \neg_G$) nor for the product logic (i.e. using $\rightarrow_P, \wedge_P, \neg_G$). For them, just consider the case $1 > a > b > 0$ to verify the inequality. We will see later on that whenever $i(a,b) \neq n(t(a, n(b))$ then under the fuzzy semantics, $\forall R.C$ is not equivalent to $\neg \exists R. \neg C$.

Fuzzy implication can also be used to determine the degree of subset relationship between two fuzzy subsets $A$ and $B$ over $X$. Indeed, we define the *degree of subsumption* between $A$ and $B$, denoted $A \rightarrow B$, as $\inf_{x \in X} i(A(x), B(x))$, where $i$ is an implication function. Note that if $\forall x \in [0,1]$, $A(x) \leq B(x)$ holds then $A \rightarrow B$ evaluates to 1. Of course, it may be that $A \rightarrow B$ evaluates to a value $0 < v < 1$ as well.

We conclude the discussion on fuzzy implication by noting that we have the following inferences: assume $a \geq n$ and $i(a,b) \geq m$. Then

- under Kleene-Dienes implication we infer that if $n > 1 - m$ then $b \geq m$. Indeed, from $i(a,b) = \max(1-a, b) \geq m$, either $1 - a \geq m$ or $b \geq m$. But $a \geq n$, so $1 - a \geq m$ implies $1 - m \geq a \geq n > 1 - m$, a contradiction. Therefore, $b \geq m$ must hold. This has been used in [29].

- under residuum based implication w.r.t. a t-norm $t$, we infer that $b \geq t(n,m)$. Indeed, from $i(a,b) = \sup\{c \colon t(a,c) \leq b\} \geq m$ and $a \geq n$ we have $t(n,m) \leq t(n,c) \leq t(a,c) \leq b$.

A (binary) *fuzzy relation* $R$ over two crisp sets $X$ and $Y$ is a function $R \colon X \times Y \rightarrow [0,1]$. The *inverse* of $R$ is the function $R^{-1} \colon Y \times X \rightarrow [0,1]$ with membership function $R^{-1}(y,x) = R(x,y)$, for every $x \in X$ and $y \in Y$. The *composition* of two fuzzy relations $R_1: X \times Y \to [0,1]$ and $R_2: Y \times Z \to [0,1]$ is defined as $(R_1 \circ R_2)(x,z) = \sup_{y \in Y} t(R_1(x,y), R_2(y,z))$, where $t$ is a t-norm. A fuzzy relation $R$ is said to be *transitive* iff $(R \circ R)(x,z) \leq R(x,z)$.

We conclude this part with fuzzy modifiers. Fuzzy modifiers applies to fuzzy sets to change their membership function. Well known examples are modifiers like `very`, `more_or_less`, `slightly`, etc. These allow us to define fuzzy sets like `very(High)` and `slightly(Mature)`. Formally, a *modifier*, $f_m$, is a function $f_m: [0,1] \to [0,1]$. For instance, we may define $\mathtt{very}(x) = x^2$, while define $\mathtt{slightly}(x) = \sqrt{x}$.

In the following, we use $\wedge, \vee, \neg$ and $\to$ in infix notation, in place of a t-norm $t$, s-norm $s$, negation $n$ and implication operator $i$.

## 3.2 Fuzzy $\mathcal{SHOIN}(\mathtt{D})$

In this section we give syntax and semantics of fuzzy $\mathcal{SHOIN}(\mathtt{D})$, using the fuzzy operators defined in the previous section. We generalize the semantics given in [13, 29, 32].

### 3.2.1 Syntax

We have seen that $\mathcal{SHOIN}(\mathtt{D})$ allows to reason with concrete data types, such as strings and integers using so-called concrete domains. In our fuzzy approach, concrete domains may be based on fuzzy sets as well.

**Concrete fuzzy domain.** A *concrete fuzzy domain* is a pair $\langle \Delta_{\mathtt{D}}, \Phi_{\mathtt{D}} \rangle$, where $\Delta_{\mathtt{D}}$ is an interpretation domain and $\Phi_{\mathtt{D}}$ is the set of concrete fuzzy domain predicates $d$ with a predefined arity $n$ and an interpretation $d^{\mathtt{D}}: \Delta_{\mathtt{D}}^n \to [0,1]$, which is a $n$-ary fuzzy relation over $\Delta_{\mathtt{D}}$.

For instance, as for $\mathcal{SHOIN}(\mathtt{D})$, the predicate $\leq_{\mathtt{18}}$ may be an unary crisp predicate over the natural numbers denoting the set of integers smaller or equal to 18, i.e. $\leq_{\mathtt{18}}: \mathtt{Natural} \to [0,1]$ and

$$\leq_{\mathtt{18}}(x) = \begin{cases} 1 & \text{if } x \leq \mathtt{18} \\ 0 & \text{otherwise .} \end{cases}$$

So,

$$\mathtt{Minor} = \mathtt{Person} \sqcap \exists \mathtt{age}. \leq_{\mathtt{18}} \quad (1)$$

defines a person, whose age is less or equal 18, i.e. it defines a minor.

On the other hand, concerning non crisp fuzzy domain predicates, we recall that in fuzzy set theory and practice there are many membership functions for fuzzy sets membership specification. However, the triangular, the trapezoidal, the $L$-function and the $R$-function are simple, yet most frequently used to specify membership degrees. The functions are defined over the set of non-negative reals $\mathbb{R}^+ \cup \{0\}$. The *trapezoidal function*, $trz(x,a,b,c,d)$, is defined as follows: let

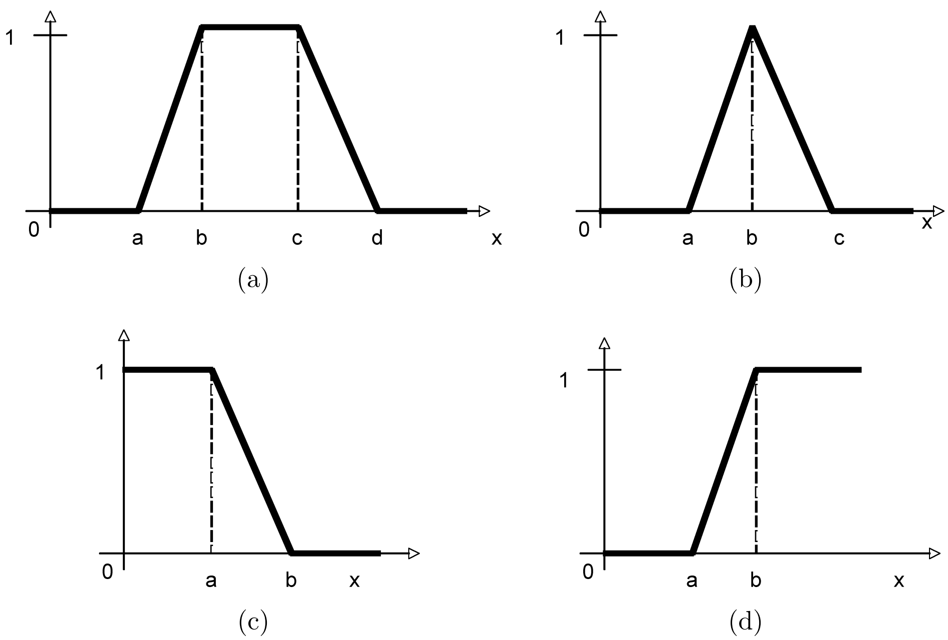


Figure 1: (a) Trapezoidal function; (b) Triangular function; (c) $L$-function; (d) $R$-function

$a < b \leq c < d$ be rational numbers then (see Figure 1)

$$trz(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ (x-a)/(b-a) & \text{if } x \in [a, b] \\ 1 & \text{if } x \in [b, c] \\ (d-x)/(d-c) & \text{if } x \in [c, d] \\ 0 & \text{if } x \geq d \ . \end{cases}$$

A *triangular function*, $tri(x; a, b, c)$, is such that (see Figure 1)

$$tri(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ (x-a)/(b-a) & \text{if } x \in [a, b] \\ (c-x)/(c-b) & \text{if } x \in [b, c] \\ 0 & \text{if } x \geq c \ . \end{cases}$$

Note that $tri(x; a, b, c) = trz(x; a, b, b, c)$.
The $L$-function is defined as (see Figure 1)

$$L(x; a, b) = \begin{cases} 1 & \text{if } x \leq a \\ (b-x)/(b-a) & \text{if } x \in [a, b] \\ 0 & \text{if } x \geq b \ . \end{cases}$$

Finally, the $R$-function is defined as (see Figure 1)

$$R(x;a,b) = \begin{cases} 0 & \text{if } x \leq a \\ (x-a)/(b-a) & \text{if } x \in [a,b] \\ 1 & \text{if } x \geq b\ . \end{cases}$$

Using these functions, we may then define, for instance, $\texttt{Young}\colon \texttt{Natural} \to [0,1]$ to be a fuzzy concrete predicate over the natural numbers denoting the degree of youngness of a person's age. The concrete fuzzy predicate `Young` may be defined as

$$\texttt{Young}(x) = L(x;10,30)\ .$$

So,

$$\texttt{YoungPerson} = \texttt{Person} \sqcap \exists \texttt{age}.\texttt{Young} \tag{2}$$

will denote a young person.

**Modifiers.** We allow modifiers in fuzzy $\mathcal{SHOIN}(\mathtt{D})$. Fuzzy modifiers, like `very`, `more_or_less` and `slightly`, apply to fuzzy sets to change their membership function. Formally, a *modifier* is a function $f_m\colon [0,1] \to [0,1]$. For instance, we may define $\texttt{very}(x) = x^2$, while define $\texttt{slightly}(x) = \sqrt{x}$. Modifiers has been considered, for instance, in [13, 32]. From a syntax point of view, if $\mathtt{M}$ is a new alphabet for modifier symbols, $m \in \mathtt{M}$ is a modifier and $C$ is a $\mathcal{SHOIN}(\mathtt{D})$ concept, then $m(C)$ is fuzzy $\mathcal{SHOIN}(\mathtt{D})$ concept as well. For instance, by referring to Example 1, we may define the concept of sports car as the concept

$$\texttt{SportsCar} = \texttt{Car} \sqcap \exists \texttt{speed}.\texttt{very}(\texttt{High})\ , \tag{3}$$

where `very` is a concept modifier, with membership function $\texttt{very}(x) = x^2$, and `High` is a fuzzy concrete predicate over the domain of speed expressed in kilometres per hour and may be defined as

$$\texttt{High}(x) = R(x;80,250)\ .$$

Similarly, we may represent "Calla is a very large, long white flower on thick stalks" as

$$\begin{array}{rcl} \texttt{Calla} & = & \texttt{Flower} \sqcap (\exists \texttt{hasSize}.\texttt{very}(\texttt{Large})) \sqcap (\exists \texttt{hasPetalWidth}.\texttt{Long}) \sqcap \\ & & \sqcap (\exists \texttt{hasColour}.\texttt{White}) \sqcap (\exists \texttt{hasStalks}.\texttt{Thick})\ , \end{array}$$

where `Large`, `Long` and `Thick` are fuzzy concrete predicates.

In summary, the syntax of fuzzy $\mathcal{SHOIN}(\mathtt{D})$ concepts is as follows:

$$\begin{array}{rcl} C & \longrightarrow & \top \mid \bot \mid A \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \neg C \mid m(C) \\ & & \forall R.C \mid \exists R.C \mid (\geq n\ S) \mid (\leq n\ S) \mid \{a_1, \ldots, a_n\} \mid \\ & & (\geq n\ T) \mid (\leq n\ T) \mid \forall T_1, \ldots, T_n.D \mid \exists T_1, \ldots, T_n.D \\ & & \\ D & \longrightarrow & d \mid \{c_1, \ldots, c_n\} \end{array}$$

Concerning axioms and assertions, similarly to [29], we introduce fuzzy axioms. Let be $n \in (0,1]$.

**Fuzzy RBox.** A fuzzy *RBox* $\mathcal{R}$ is a finite set of $\mathcal{SHOIN}(\mathtt{D})$ transitivity axioms $\mathtt{trans}(R)$ and *fuzzy role inclusion axioms* of the form $\langle \alpha \geq n \rangle$, $\langle \alpha \leq n \rangle$, $\langle \alpha > n \rangle$ and $\langle \alpha > n \rangle$, where $\alpha$ is a $\mathcal{SHOIN}(\mathtt{D})$ role inclusion axiom.

**Fuzzy TBox.** A fuzzy *TBox* $\mathcal{T}$ consists of a finite set of *fuzzy concept inclusion axioms* of the form $\langle \alpha \geq n \rangle$, $\langle \alpha \leq n \rangle$, $\langle \alpha > n \rangle$ and $\langle \alpha < n \rangle$ where $\alpha$ is a $\mathcal{SHOIN}(\mathtt{D})$ concept inclusion axiom;

**Fuzzy ABox.** A fuzzy *ABox* $\mathcal{A}$ consists of a finite set of *fuzzy concept* and *fuzzy role assertion axioms* of the form $\langle \alpha \geq n \rangle$, $\langle \alpha \leq n \rangle$, $\langle \alpha > n \rangle$, or $\langle \alpha < n \rangle$, where $\alpha$ is a $\mathcal{SHOIN}(\mathtt{D})$ concept or role assertion. As for the crisp case, $\mathcal{A}$ may also contain a finite set of individual (in)equality axioms $a \approx b$ and $a \not\approx b$, respectively.

For instance, $\langle a{:}C \geq 0.1 \rangle$, $\langle (a,b){:}R \leq 0.3 \rangle$, $\langle R \sqsubseteq S \geq 0.4 \rangle$, or $\langle C \sqsubseteq D \leq 0.6 \rangle$ are fuzzy axioms. Informally, from a semantics point of view, a fuzzy axiom $\langle \alpha \leq n \rangle$ constrains the membership degree of $\alpha$ to be less or equal to $n$ (similarly for $\geq, >, <$). As a consequence,

$$\langle \mathtt{jim}{:}\mathtt{YoungPerson} \geq 0.2 \rangle \ ,$$

i.e. $\langle \mathtt{jim}{:}\mathtt{Person} \sqcap \exists \mathtt{age}.\mathtt{Young} \geq 0.2 \rangle$, dictates that $\mathtt{jim}$ is a $\mathtt{YoungPerson}$ with degree at least 0.2.

On the other hand, a fuzzy concept inclusion axiom of the form

$$\langle C \sqsubseteq D \geq n \rangle$$

dictates that the subsumption degree between $C$ and $D$ is at least $n$.

**Fuzzy knowledge base.** A $\mathcal{SHOIN}(\mathtt{D})$ *fuzzy knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ consists of a fuzzy TBox $\mathcal{T}$, a fuzzy RBox $\mathcal{R}$, and a fuzzy ABox $\mathcal{A}$.

### 3.2.2 Semantics

The semantics extends [29]. The main idea is that concepts and roles are interpreted as fuzzy subsets of an interpretation's domain. Therefore, $\mathcal{SHOIN}(\mathtt{D})$ axioms, rather being satisfied (true) or unsatisfied (false) in an interpretation, become a degree of truth in $[0,1]$.

**Fuzzy interpretation.** A *fuzzy interpretation* $\mathcal{I}$ with respect to a concrete domain $\mathtt{D}$ is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (called the *domain*), disjoint from $\Delta_{\mathtt{D}}$, and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns

- to each abstract concept $C \in \mathtt{C}$ a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$;
- to each abstract role $R \in \mathtt{R}_a$ a function $R^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$;

- to each abstract functional role $R \in \mathtt{R}_a$ a partial function $R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$ such that for all $x \in \Delta^{\mathcal{I}}$ there is an unique $y \in \Delta^{\mathcal{I}}$ on which $R^{\mathcal{I}}(x,y)$ is defined;
- to each abstract individual $a \in \mathtt{I}_a$ an element in $\Delta^{\mathcal{I}}$;
- to each concrete individual $c \in \mathtt{I}_c$ an element in $\Delta_{\mathtt{D}}$;
- to each concrete role $T \in \mathtt{R}_c$ a function $R^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta_{\mathtt{D}} \to [0,1]$;
- to each concrete functional role $T \in \mathtt{R}_c$ a partial function $t^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta_{\mathtt{D}} \to [0,1]$ such that for all $x \in \Delta^{\mathcal{I}}$ there is an unique $v \in \Delta_{\mathtt{D}}$ on which $T^{\mathcal{I}}(x,v)$ is defined;
- to each modifier $m \in \mathtt{M}$ the modifier function $f_{\mathtt{m}}\colon [0,1] \to [0,1]$;
- to each $n$-ary concrete fuzzy predicate $d$ the fuzzy relation $d^{\mathtt{D}}\colon \Delta_{\mathtt{D}}^n \to [0,1]$.

The mapping $\cdot^{\mathcal{I}}$ is extended to concepts and roles as specified in the following table (where $x, y \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathtt{D}}$):

$$
\begin{array}{rcl}
\top^{\mathcal{I}}(x) & = & 1 \\
\bot^{\mathcal{I}}(x) & = & 0 \\
(C_1 \sqcap C_2)^{\mathcal{I}}(x) & = & {C_1}^{\mathcal{I}}(x) \wedge {C_2}^{\mathcal{I}}(x) \\
(C_1 \sqcup C_2)^{\mathcal{I}}(x) & = & {C_1}^{\mathcal{I}}(x) \vee {C_2}^{\mathcal{I}}(x) \\
(\neg C)^{\mathcal{I}}(x) & = & \neg C^{\mathcal{I}}(x)) \\
(m(C))^{\mathcal{I}}(x) & = & f_{\mathtt{m}}(C^{\mathcal{I}}(x)) \\
(\forall R.C)^{\mathcal{I}}(x) & = & \inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \to C^{\mathcal{I}}(y) \\
(\exists R.C)^{\mathcal{I}}(x) & = & \sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y) \\
(\geq n\ S)^{\mathcal{I}}(x) & = & \sup_{\substack{\{y_1,\ldots,y_n\} \subseteq \Delta^{\mathcal{I}} \\ |\{y_1,\ldots,y_n\}| = n}} \bigwedge_{i=1}^n S^{\mathcal{I}}(x,y_i) \\
(\leq n\ S)^{\mathcal{I}}(x) & = & \neg(\geq n+1\ S)^{\mathcal{I}}(x) \\
\{a_1,\ldots,a_n\}^{\mathcal{I}}(x) & = & \bigvee_{i=1}^n {a_i}^{\mathcal{I}} = x \\
d(v) & = & d^{\mathtt{D}}(v) \\
\{c_1,\ldots,c_n\}^{\mathcal{I}}(v) & = & \bigvee_{i=1}^n {c_i}^{\mathcal{I}} = v \\
(\forall T_1,\ldots,T_n.D)^{\mathcal{I}}(x) & = & \inf_{y_1,\ldots,y_n \in \Delta_{\mathtt{D}}{}^{\mathcal{I}}} (\bigwedge_{i=1}^n {T_i}^{\mathcal{I}}(x,y_i)) \to D^{\mathcal{I}}(y_1,\ldots,y_n) \\
(\exists T_1,\ldots,T_n.D)^{\mathcal{I}}(x) & = & \sup_{y_1,\ldots,y_n \in \Delta_{\mathtt{D}}{}^{\mathcal{I}}} (\bigwedge_{i=1}^n {T_i}^{\mathcal{I}}(x,y_i)) \wedge D^{\mathcal{I}}(y_1,\ldots,y_n) \\
(S^-)^{\mathcal{I}}(x,y) & = & S^{\mathcal{I}}(y,x)\ .
\end{array}
$$

We comment briefly some points. The semantics of $\exists R.C$

$$
(\exists R.C)^{\mathcal{I}}(d) \quad = \quad \sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \wedge C^{\mathcal{I}}(y)
$$

is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x,y) \wedge F_C(y)$ (where $F$ is the obvious translation of roles and concepts into First-Order Logic - FOL) and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Similarly,

$$
(\forall R.C)^{\mathcal{I}}(x) \quad = \quad \inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \to C^{\mathcal{I}}(y)
$$

is related to the open first order formula $\forall y.F_R(x,y) \to F_C(y)$, where the universal quantifier $\forall$ is viewed as a conjunction over the elements of the domain. However, as we already pointed out in Section 3.1, unlike the classical case, in general we do not have that $(\forall R.C)^{\mathcal{I}} = (\neg\exists R.\neg C)^{\mathcal{I}}$. For instance, it holds in Lukasiewicz logic, but not in Gödel logic. Also interesting is that (see [12]) the axiom $\top \sqsubseteq \neg(\forall R.A) \sqcap (\neg\exists R.\neg A)$ has no classical model. However, in [12] it is shown that in Gödel logic it has no finite model, but has an infinite model.

Another point concerns the semantics of number restrictions. The semantics of the concept $(\geq n\ S)$

$$(\geq n\ S)^{\mathcal{I}}(x) \quad = \quad \sup_{\substack{\{y_1,\ldots,y_n\} \subseteq \Delta^{\mathcal{I}} \\ |\{y_1,\ldots,y_n\}| = n}} \textstyle\bigwedge_{i=1}^{n} S^{\mathcal{I}}(x,y_i)$$

is the result of viewing $(\geq n\ S)$ as the open first order formula

$$\exists y_1,\ldots,y_n. \bigwedge_{i=1}^{n} S(x,y_i) \wedge \bigwedge_{1 \leq i < j \leq n} y_i \neq y_j \ .$$

That is, there are at least $n$ distinct elements that satisfy to some degree $S(x,y_i)$. This guarantees us that $\exists S.\top \equiv (\geq 1\ S)$. The semantics of $(\leq n\ S)$ is defined in such a way to guarantee the classical relationship $(\leq n\ S) \equiv \neg(\geq n+1\ S)$.

An alternative definition for the $(\geq n\ S)$ and the $(\leq n\ S)$ constructs may rely on the scalar cardinality of a fuzzy set. However, we prefer to stick on the formulation, which derives directly from its FOL translation.

Finally, the mapping $\cdot^{\mathcal{I}}$ is extended to non-fuzzy axioms as specified in the following table (where $a, b \in \mathtt{I}_a$):

$$\begin{array}{rcl}
(R \sqsubseteq S)^{\mathcal{I}} & = & \inf_{x,y\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \to S^{\mathcal{I}}(x,y) \\
(T \sqsubseteq U)^{\mathcal{I}} & = & \inf_{x,y\in\Delta^{\mathcal{I}}} T^{\mathcal{I}}(x,y) \to U^{\mathcal{I}}(x,y) \\
(C \sqsubseteq D)^{\mathcal{I}} & = & \inf_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \to D^{\mathcal{I}}(x) \\
(a{:}C)^{\mathcal{I}} & = & C^{\mathcal{I}}(a^{\mathcal{I}}) \\
((a,b){:}R)^{\mathcal{I}} & = & R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \ .
\end{array}$$

Note here that e.g. the semantics of a concept inclusion axiom $C \sqsubseteq D$ is derived directly from its FOL translation, which is of the form $\forall x.F_C(x) \to F_D(x)$. This definition is novel and is clearly different from the approaches in which $C \sqsubseteq D$ is viewed as $\forall x.C(x) \leq D(x)$. This latter approach has the effect that the subsumption relationship is a classical $\{0,1\}$ relationship, while the former has the advantage that subsumption is determined up to a certain degree in $[0,1]$.

**Satisfiability.** The notion of *satisfiability* of a fuzzy axiom $E$ by a fuzzy interpretation $\mathcal{I}$, denoted $I \models E$, is defined as follows: $\mathcal{I} \models \mathtt{trans}(R)$, iff $\forall x,y \in \Delta^{\mathcal{I}}.R^{\mathcal{I}}(x,y) = \sup_{z\in\Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,z) \wedge R^{\mathcal{I}}(z,y)$. $I \models \langle \alpha \geq n \rangle$, where $\alpha$ is a role inclusion or concept inclusion axiom, iff $\alpha^{\mathcal{I}} \geq n$. Similarly, for the other relations $\leq, <$ and $>$. $I \models \langle \alpha \geq n \rangle$, where $\alpha$ is a concept or a role assertion axiom, iff $\alpha^{\mathcal{I}} \geq n$. Similarly, for the other relations $\leq, <, >$. Finally, $\mathcal{I} \models a \approx b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$ and $\mathcal{I} \models a \not\approx b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

For a set of fuzzy axioms $\mathcal{E}$, we say that $I$ *satisfies* $\mathcal{E}$ iff $I$ satisfies each element in $\mathcal{E}$. If $I \models E$ (resp. $I \models \mathcal{E}$) we say that $I$ is a *model* of $E$ (resp. $\mathcal{E}$). $I$ *satisfies* (is a *model* of) a fuzzy knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, denoted $I \models \mathcal{K}$, iff $I$ is a model of each component $\mathcal{T}, \mathcal{R}$ and $\mathcal{A}$, respectively.

**Logical consequence.** A fuzzy axiom $E$ is a *logical consequence* of a knowledge base $\mathcal{K}$, denoted $\mathcal{K} \models E$ iff every model of $\mathcal{K}$ satisfies $E$.

The interesting point is that according to our semantics, e.g. a minor is a young person to a certain degree and is obtained without explicitly mentioning it. This inference can not be achieved in classical $\mathcal{SHOIN}(\mathtt{D})$. Similarly, by referring to Example 1, we will have that the car `tt` will be a sports car to a certain degree. Therefore, unlike Example 1, `tt` is now likely a sport car, *as it should be*. The following two examples highlight these points.

**Example 2** *Let us consider Example 1, where all axioms of the TBox and ABox are asserted with degree* 1*, i.e. are of the form* $\langle \alpha \geq 1 \rangle$*. We replace the definition of* `SportsCar` *with Definition (3). Then we have that (interpreting conjunction as* min*)*

$$\mathcal{K} \models \langle \mathtt{SportsCar} \sqsubseteq \mathtt{Car} \geq 1 \rangle \qquad \mathcal{K} \models \langle \mathtt{mgb}{:}\mathtt{SportsCar} \leq 0.25 \rangle$$
$$\mathcal{K} \models \langle \mathtt{enzo}{:}\mathtt{SportsCar} \geq 1 \rangle \qquad \mathcal{K} \models \langle \mathtt{tt}{:}\mathtt{SportsCar} \geq 0.82 \rangle \ .$$

*Note how the maximal speed limit of the* `mgb` *car* ($\leq_{\mathtt{170km/h}}$) *induces an upper limit,* 0.25*, of the membership degree. Neither this inference is possible in classical* $\mathcal{SHOIN}(\mathtt{D})$*, nor the one involving* `tt`*.*

**Example 3** *Consider the knowledge base* $\mathcal{K}$ *with Definitions (1) and (2). Then under Lukasiewicz logic we have that (see [31])*

$$\mathcal{K} \models \langle \mathtt{Minor} \sqsubseteq \mathtt{YoungPerson} \geq 0.2 \rangle \ ,$$

*which is a relationship not captured with classical* $\mathcal{SHOIN}(\mathtt{D})$*.*

**Best truth value bound.** Finally, given $\mathcal{K}$ and an axiom $\alpha$, where $\alpha$ is neither a transitivity axiom, nor an individual (in) equality axiom, it is of interest to compute $\alpha$'s best lower and upper degree value bounds. The *greatest lower bound* of $\alpha$ w.r.t. $\mathcal{K}$ (denoted $glb(\mathcal{K}, \alpha)$) is

$$glb(\mathcal{K}, \alpha) = \sup\{n : \mathcal{K} \models \langle \alpha \geq n \rangle\} \ ,$$

while the *least upper bound* of $\alpha$ with respect to $\mathcal{K}$ (denoted $lub(\mathcal{K}, \alpha)$ is

$$lub(\mathcal{K}, \alpha) = \inf\{n : \mathcal{K} \models \langle \alpha \leq n \rangle\} \ ,$$

where $\sup \emptyset = 0$ and $\inf \emptyset = 1$. Determining the *lub* and the *glb* is called the *Best Degree Bound* (BDB) problem. For instance, the entailments in Examples 2 and 3 are the best possible degree bounds. Furthermore, note that,

$$lub(\Sigma, a{:}C) = \neg glb(\Sigma, a{:}\neg C) \ , \tag{4}$$

i.e. the *lub* can be determined through the *glb* (and vice-versa). Similarly, $lub(\Sigma, (a,b){:}R) = \neg glb(\Sigma, a{:}\neg\exists R.\{b\})$ holds. Also, note that, $\Sigma \models \langle \alpha \geq n \rangle$ iff $glb(\Sigma, \alpha) \geq n$, and similarly $\Sigma \models \langle \alpha \leq n \rangle$ iff $lub(\Sigma, \alpha) \leq n$ hold.

Concerning the entailment problem, it is quite easily verified that, as for the crisp case, the entailment problem can be reduced to the unsatisfiability problem:

$$\begin{array}{lll} \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle \models \langle \alpha \geq n \rangle & \text{iff} & \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\langle \alpha < n \rangle\} \rangle \text{ is not satisfiable} \\ \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle \models \langle \alpha \leq n \rangle & \text{iff} & \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \cup \{\langle \alpha > n \rangle\} \rangle \text{ is not satisfiable} \ . \end{array}$$

### 3.3 Reasoning

Unfortunately, from a computational point of view, no calculus exists yet checking satisfiability of fuzzy $\mathcal{SHOIN}(\mathtt{D})$ knowledge bases. [13, 32] report a calculus for the case of $\mathcal{ALC}$ [26] (with concept constructors $\top, \bot, \neg, \sqcap, \sqcup, \forall, \exists$) with modifiers and simple TBox, with $\min, \max$ and $\rightarrow_{KD}$ connectives. No indication for the BDB problem is given. [27, 29] reports a calculus for $\mathcal{ALC}$ and simple TBox, with $\min, \max$ and $\rightarrow_{KD}$ connectives and addresses the BDB problem and, [30] shows how the satisfiability problem and the BDB problem can be reduced to classical $\mathcal{ALC}$ and, thus, can be resolved by means of a tools like FACT and RACER. However, despite these negative results, recently [31] reports a calculus for $\mathcal{ALC}(\mathtt{D})$ whenever the connectives, the modifiers and the concrete fuzzy predicates are representable as a bounded Mixed Integer Program. For instance, Lukasiewicz logic satisfies these conditions as well as the membership functions for concrete fuzzy predicates we have presented in this paper. Additionally, modifiers should be a combination of linear functions. In that case the calculus consists of a set of constraint propagation rules and an invocation to an oracle for bounded Mixed Integer Programming. But, indeed, the computational aspect is definitely a point that has to be addressed in forthcoming works.

## 4 Related work

Several ways of extending DL using the theory of fuzzy logic have been proposed in the literature. The first work is due to Yen [33] who considered a sub-language of $\mathcal{ALC}$, $\mathcal{FL}^-$ [5]. However, it already informally talks about the use of modifiers and concrete domains. Though, the unique reasoning facility, the subsumption test, is a crisp yes/no question. Tresp [32] considered fuzzy $\mathcal{ALC}$ extended with a special form of modifiers, which are a combination of two linear functions. min, max and $1-x$ membership functions has been considered and a sound and complete reasoning algorithm testing the subsumption relationship has been presented. Similar to our approach, a linear programming oracle is needed. Assertional reasoning has been considered by Straccia [27, 28, 29], where fuzzy assertion axioms have been allowed in fuzzy $\mathcal{ALC}$ (with min, max and $1-x$ functions), concept modifiers are not allowed however ([28] reports a four-valued variant of fuzzy $\mathcal{ALC}$). He also introduced the BDB problem and provided a sound and complete reasoning algorithm based on completion rules ([30] provides a translation of fuzzy $\mathcal{ALC}$ into classical $\mathcal{ALC}$). For an application see [24]. In the same spirit [13] extend Straccia's fuzzy $\mathcal{ALC}$ with concept modifiers of the form $f_m(x) = x^\beta$, where $\beta > 0$. A sound and complete reasoning algorithm for the graded subsumption problem, based on completion rules, is presented. Finally, [25] start addressing the issue of alternative semantics of quantifiers in fuzzy $\mathcal{ALC}$ (without the assertional component). No reasoning algorithm is given. Concerning [31], we already addressed it in the previous section. Finally, [12] considers $\mathcal{ALC}$ under arbitrary t-norm and reports, among others, a procedure deciding $\models \langle C \sqsubseteq D \geq 1\rangle$ and deciding whether $\langle C \sqsubseteq D \geq 1\rangle$ is satisfiable, by a reduction to the propositional BL logic.

# 5 Conclusions and outlook

We have presented a fuzzy extension of $\mathcal{SHOIN}(\mathtt{D})$ showing that its representation and reasoning capabilities go clearly beyond classical $\mathcal{SHOIN}(\mathtt{D})$. Interestingly, we allow modifiers, fuzzy concrete domain predicates and fuzzy axioms to appear in a $\mathcal{SHOIN}(\mathtt{D})$ knowledge base and the entailment and the subsumption relationship hold to a certain degree. To the best of our knowledge, no other work has yet extended the semantics to $\mathcal{SHOIN}(\mathtt{D})$ in such a way. The argument supporting the necessity of such an extension relies on the fact that vague concepts are abundant in human knowledge and, thus, appear *likely* in Web content.

The main direction for future work involves the computational aspect. Currently, we are addressing the fundamental issue to develop a calculus for reasoning within $\mathcal{SHOIN}(\mathtt{D})$, extending [31].

Another direction is in extending fuzzy $\mathcal{SHOIN}(\mathtt{D})$ with *fuzzy quantifiers* (see [19, 20] for an overview on fuzzy quantifiers), where the $\forall$ and $\exists$ quantifiers are replaced with fuzzy quantifiers like `most`, `some`, `usually` and the like (see [25] for a preliminary work in this direction). This allows to define concepts like

$$\begin{array}{l}\mathtt{TopCustomer} = \mathtt{Customer} \sqcap (\mathtt{Usually})\mathtt{buys.ExpensiveItem} \\ \mathtt{ExpensiveItem} = \mathtt{Item} \sqcap \exists \mathtt{price.High}\ .\end{array}$$

Here, the fuzzy quantifier `Usually` replaces the classical quantifier $\forall$ and `High` is a fuzzy concrete predicate. Fuzzy quantifiers can be applied to inclusion axioms as well, allowing to express, e.g.

$$(\mathtt{Most})\mathtt{Bird} \sqsubseteq \mathtt{FlyingObject}\ .$$

Here the fuzzy quantifier `Most` replaces the classical universal quantifier $\forall$ assumed in the inclusion axioms. The above axiom allows to state that most birds fly.

Ultimately, we believe that the fuzzy extension of $\mathcal{SHOIN}(\mathtt{D})$ is of great interest to the Semantic Web community, as it allows to express naturally a wide range of concepts of actual domains, for which a classical $\mathcal{SHOIN}(\mathtt{D})$ representation is unsatisfactory.

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