Towards a Crisp Representation of Fuzzy Description Logics under Łukasiewicz Semantics

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Abstract. Classical ontologies are not suitable to represent imprecise nor uncertain pieces of information. Fuzzy Description Logics were born to represent the former type of knowledge, but they require an appropriate fuzzy language to be agreed and an important number of available resources to be adapted. An alternative is to use classical ontologies to represent fuzzy ontologies. To date, all of the work in this direction has restricted to the Zadeh family of fuzzy operators. In this paper, we generalize existing proposals and propose a reasoning preserving procedure to obtain a crisp representation for a fuzzy extension of the logic $\mathcal{ALCHOI}$ under Łukasiewicz semantics. This reduction makes possible to reuse a crisp representation language as well as currently available reasoners under crisp semantics.

1 Introduction

Description Logics (DLs for short) [1] are a family of logics for representing structured knowledge which have proved to be very useful as ontology languages. For instance, the standard Web Ontology Language OWL [2] can be divided in three levels, namely OWL Full, OWL DL and OWL Lite, with $\mathcal{SHOIN}(\mathbf{D})$ and $\mathcal{SHIF}(\mathbf{D})$ DLs being the subjacent formalisms of OWL DL and OWL Lite.

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real-world domains. Since fuzzy logic is a suitable formalism to handle these types of knowledge, several fuzzy extensions of DLs can be found in the literature (see [3] for an overview).

Defining a fuzzy DL brings about that crisp standard languages would no longer be appropriate, new fuzzy languages should be used and hence the large number of resources available should be adapted to the new framework, requiring an important effort. An alternative is to represent fuzzy DLs using crisp DLs and to reason using their (crisp) reductions. This approach has several advantages:

- There would be no need to agree a new standard fuzzy language, but every developer could use its own language as long as he implements the reduction that we describe.

[⋆] The work of F. Bobillo has been partially supported by the Spanish Ministerio de Educación y Ciencia under project TIN2006-15041-C04-01 and a FPU scholarship.

A. An et al. (Eds.): ISMIS 2008, LNAI 4994, pp. 309–318, 2008.

- We will continue using standard languages with a lot of resources available, so the need (and cost) of adapting them to the new fuzzy language is avoided.
- We will use existing crisp reasoners. We do not claim that reasoning will be more efficient, but it supposes an easy alternative to support early reasoning in future fuzzy languages. In fact, nowadays there is no reasoner fully supporting a fuzzy extension of OWL DL under Łukasiewicz semantics[1].

Under this approach an immediate practical application of fuzzy ontologies is feasible, because of its tight relation with already existing languages and tools which have proved their validity.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory [3], the representation of them using crisp description logics has not received such attention. The first efforts in this direction are due to Straccia, who considered fuzzy $\mathcal{ALCH}$ [4] and fuzzy $\mathcal{ALC}$ with truth values taken from an uncertainty lattice [5]. Bobillo et al. widened the representation to $\mathcal{SHOIN}$ [7]. Stoilos et al. extended this work with some additional role axioms [8]. Finally, Bobillo et al. proposed a full crisp representation of $\mathcal{SROIQ}$, and optimized the process by reducing the size of the resulting ontology [9]. However, from a semantics point of view, these works restrict themselves to the Zadeh family of fuzzy operators, which has some limitations (see [6,7] for some counter-intuitive examples). This paper provides a crisp representation for a fuzzy DL under Łukasiewicz semantics, which generalizes Zadeh family.

The remainder of this work is organized as follows. Section 2 overviews some important results in fuzzy set theory. Section 3 describes a fuzzy extension of $\mathcal{ALCH}$ under Łukasiewicz semantics. Section 4 depicts a reduction into crisp $\mathcal{ALCH}$ and extends the result to $\mathcal{ALCHOI}$. Finally, Section 5 sets out some conclusions and ideas for future work.

2 Fuzzy Set Theory

Fuzzy set theory and fuzzy logic were proposed by Zadeh [10] to manage imprecise and vague knowledge. While in classical set theory elements either belong to a set or not, in fuzzy set theory elements can belong to a degree of certainty. More formally, let X be a set of elements called the reference set. A fuzzy subset A of X, is defined by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns any $x \in X$ to a value in the real interval between 0 and 1. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which x can be considered as an element of X.

For every $\alpha \in [0,1]$, the *α-cut* of a fuzzy set A is defined as the set such as its elements belong to A with degree at least α, i.e. $A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$. Similarly, the *strict α-cut* is defined as $A_{\alpha+} = \{x \mid \mu_A(x) > \alpha\}$. Notice that these sets are crisp.

All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication set operations are performed by a t-norm function t, a t-conorm function u, a negation function c and an implication function i,

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[1] The *fuzzyDL* reasoner (see Straccia's Web page) supports fuzzy OWL-Lite so far.

respectively. For a definition of these functions we refer the reader to [11]. We will mention throughout this paper two families of fuzzy operators, Zadeh and Łukasiewicz, which are defined as follows:

Family	t-norm $t(\alpha, \beta)$	t-conorm $u(\alpha, \beta)$	negation $c(\alpha)$	implication $i(\alpha, \beta)$
Zadeh	$\min\{\alpha, \beta\}$	$\max\{\alpha, \beta\}$	$1 - \alpha$	$\max\{1 - \alpha, \beta\}$
Łukasiewicz	$\max\{\alpha + \beta - 1, 0\}$	$\min\{\alpha + \beta, 1\}$	$1 - \alpha$	$\min\{1, 1 - \alpha + \beta\}$

Let $\otimes$, $\oplus$, $\ominus$ and $\Rightarrow$ denote the Łukasiewicz family of fuzzy operators (t-norm, t-conorm, negation and implication, respectively) and let $\wedge$, $\vee$, $\neg$ and $\rightarrow$ denote the Zadeh family. Interestingly, using the Łukasiewicz family it is possible to represent the operators of Zadeh family:

$$\begin{array}{cc} \neg\alpha = \ominus\alpha & \alpha \wedge \beta = \alpha \otimes (\alpha \Rightarrow \beta) \\ \alpha \vee \beta = \neg((\neg\alpha) \wedge (\neg\beta)) & \alpha \rightarrow \beta = (\neg\alpha) \vee \beta \end{array}$$

3 Fuzzy $\mathcal{ALCH}$

In this section we define a fuzzy extension of the DL $\mathcal{ALCH}$ where concepts denote fuzzy sets of individuals and roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. Then, we will restrict ourselves to the Łukasiewicz family of fuzzy operators.

The following definition is based on the fuzzy DL presented in [12].

Syntax. Fuzzy $\mathcal{ALCH}$ assumes three alphabets of symbols, for concepts, roles and individuals. The concepts of the language (denoted C or D) can be built inductively from atomic concepts (A), atomic roles (R), top concept $\top$ and bottom concept $\bot$ according to the following syntax rule:

$$C, D \rightarrow A \mid \top \mid \bot \mid C \sqcap D \mid C \sqcup D \mid \neg C \mid \forall R.C \mid \exists R.C$$

Notice that the syntax is the same as in the crisp case.

A fuzzy Knowledge Base (KB) comprises two parts: the extensional knowledge, i.e. particular knowledge about some specific situation (a fuzzy Assertional Box or ABox $\mathcal{A}$ with statements about individuals) and the intensional knowledge, i.e. general knowledge about the application domain (a fuzzy Terminological Box or TBox $\mathcal{T}$ and a fuzzy Role Box or RBox $\mathcal{R}$).

In the rest of the paper we will assume $\bowtie \in \{\geq, \leq\}$, $\alpha \in (0, 1]$, $\beta \in [0, 1)$ and $\gamma \in [0, 1]$. Moreover, for every operator $\bowtie$ we define its symmetric operator $\bowtie^-$ as $\geq^- = \leq, \leq^- = \geq$.

A *fuzzy ABox* consists of a finite set of *fuzzy assertions* of the following types:

- concept assertions $\langle a : C \geq \alpha \rangle$ or $\langle a : C \leq \beta \rangle$,
- role assertions $\langle (a, b) : R \geq \alpha \rangle$.

A *fuzzy TBox* consists of fuzzy General Concept Inclusions (fuzzy GCIs), expressions of the form $\langle C \sqsubseteq D \geq \alpha \rangle$.

A *fuzzy RBox* consists of a finite set of fuzzy Role Inclusion Axioms (fuzzy RIAs) of the form $\langle R \sqsubseteq R' \geq \alpha \rangle$.

A fuzzy axiom τ is *positive* (denoted $\langle \tau \rhd \alpha \rangle$) if it is of the form $\langle \tau \geq \alpha \rangle$, and *negative* (denoted $\langle \tau \lhd \alpha \rangle$) if it is of the form $\langle \tau \leq \beta \rangle$. $\langle \tau = \alpha \rangle$ is equivalent to the pair of axioms $\langle \tau \geq \alpha \rangle$ and $\langle \tau \leq \alpha \rangle$.

Notice that negative role assertions, GCIs or RIAs are not allowed, because they correspond to negated role assertions, GCIs and RIAs respectively, which are not part of crisp $\mathcal{ALCH}$.

Note also that for the sake of clarity we are only considering inequalities of the form $\geq$ and $\leq$, but extending the language with $>$ and $<$ is not complicated.

Semantics. A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

- every individual onto an element of $\Delta^{\mathcal{I}}$,
- every concept C onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$,
- every role R onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$.

$C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) denotes the membership function of the fuzzy concept C (resp. fuzzy role R) w.r.t. $\mathcal{I}$. $C^{\mathcal{I}}(a)$ (resp. $R^{\mathcal{I}}(a, b)$) gives us the degree of being the individual a an element of the fuzzy concept C (resp. the degree of being (a, b) an element of the fuzzy role R) under the fuzzy interpretation $\mathcal{I}$.

For a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts as follows:

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\begin{aligned}
\top^{\mathcal{I}}(a) &= 1 \\
\bot^{\mathcal{I}}(a) &= 0 \\
(C \sqcap D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \otimes D^{\mathcal{I}}(a) \\
(C \sqcup D)^{\mathcal{I}}(a) &= C^{\mathcal{I}}(a) \oplus D^{\mathcal{I}}(a) \\
(\neg C)^{\mathcal{I}}(a) &= \ominus C^{\mathcal{I}}(a) \\
(\forall R.C)^{\mathcal{I}}(a) &= \inf_{b \in \Delta^{\mathcal{I}}} \{ R^{\mathcal{I}}(a, b) \Rightarrow C^{\mathcal{I}}(b) \} \\
(\exists R.C)^{\mathcal{I}}(a) &= \sup_{b \in \Delta^{\mathcal{I}}} \{ R^{\mathcal{I}}(a, b) \otimes C^{\mathcal{I}}(b) \}
\end{aligned}
$$

A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- $\langle a : C \geq \alpha \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha$,
- $\langle a : C \leq \beta \rangle$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq \beta$,
- $\langle (a, b) : R \geq \alpha \rangle$ iff $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq \alpha$,
- $\langle C \sqsubseteq D \geq \alpha \rangle$ iff $\inf_{a \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(a) \Rightarrow D^{\mathcal{I}}(a) \} \geq \alpha$,
- $\langle R \sqsubseteq R' \geq \alpha \rangle$ iff $\inf_{a, b \in \Delta^{\mathcal{I}}} \{ R^{\mathcal{I}}(a, b) \Rightarrow R'^{\mathcal{I}}(a, b) \} \geq \alpha$,

We say that $\mathcal{I}$ *satisfies* a fuzzy KB iff it satisfies each element in it. In the rest of the paper we will only consider fuzzy KB satisfiability, since (as in the crisp case) most inference problems can be reduced to it [13].

It can be easily shown that this fuzzy extension of $\mathcal{ALCH}$ is a sound extension of crisp $\mathcal{ALCH}$, because fuzzy interpretations coincide with crisp interpretations if we restrict the membership degrees to $\{0, 1\}$.

Here in after we will concentrate on Ł-$\mathcal{ALCH}$, restricting ourselves to the fuzzy operators of the Łukasiewicz family.

4 A Crisp Representation for Fuzzy Ł-$\mathcal{ALCH}$

In this section we show how to reduce a Ł-$\mathcal{ALCH}$ fuzzy KB into a crisp knowledge base (KB). We will start by presenting our reduction procedure, then we will discuss the properties of the reduction, showing that it preserves reasoning, so existing $\mathcal{ALCH}$ reasoners could be applied to the resulting KB, and illustrate the full procedure with an example.

The basic idea is to create some new crisp concepts and roles, representing the α-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new crisp elements.

4.1 Adding New Elements

U. Straccia showed [4] that, for a fuzzy KB fK, the set of the degrees which must be considered for any reasoning task is defined as $N^{fK} = X^{fK} \cup \{1 - \alpha \mid \alpha \in X^{fK}\}$, where $X^{fK} = \{0, 0.5, 1\} \cup \{\gamma \mid \langle \tau \bowtie \gamma \rangle \in fK\}$ [4]. This holds for fuzzy DLs under Zadeh semantics, but it is not true in general when other fuzzy operators are considered. Interestingly, in the case of Łukasiewicz logic it is true if we fix the number of allowed degrees.

In fact, let q be a natural number with $q \geq 1$. We assume a set of $q + 1$ allowed truth degrees in the fuzzy KB, i.e., $\mathcal{N} = \{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{(q-1)}{q}, 1\}$. The following proposition shows that, using the fuzzy operators of Łukasiewicz logic to combine two truth degrees a and b, no new degrees can appear.

Proposition 1. *Let $\frac{a}{q}, \frac{b}{q} \in \mathcal{N}$. Then, under the fuzzy operators of Łukasiewicz logic, $\ominus \frac{a}{q}, \frac{a}{q} \otimes \frac{b}{q}, \frac{a}{q} \oplus \frac{b}{q}, \frac{a}{q} \Rightarrow \frac{b}{q} \in \mathcal{N}$.*

Proof. Let us consider each of the four fuzzy operators:

- $\ominus \frac{a}{q} = 1 - \frac{a}{q} = \frac{q-a}{q}$ belongs to $\mathcal{N}$: since $a \in [0, q], (q - a) \in [0, q]$.
- $\frac{a}{q} \otimes \frac{b}{q} = \max\{\frac{a}{q} + \frac{b}{q} - 1, 0\}$. If $\frac{a}{q} + \frac{b}{q} - 1 \leq 0$, then the value of the conjunction is 0, which obviously belongs to $\mathcal{N}$. Otherwise, its value is $\frac{a+b-q}{q}$ which also belongs to $\mathcal{N}$: since $a, b \in [0, q]$ and $\frac{a}{q} + \frac{b}{q} - 1 > 0$, it follows that $(a + b - q) \in [0, q]$.
- $\frac{a}{q} \oplus \frac{b}{q} = \min\{\frac{a}{q} + \frac{b}{q}, 1\}$. If $\frac{a}{q} + \frac{b}{q} > 1$, then the value of the disjunction is 1, which obviously belongs to $\mathcal{N}$. Otherwise, its value is $\frac{a+b}{q}$ which also belongs to $\mathcal{N}$: since $a, b \in [0, q]$ and $\frac{a}{q} + \frac{b}{q} \leq 1$, it follows that $(a + b) \in [0, q]$.
- $\frac{a}{q} \Rightarrow \frac{b}{q} = \min\{1 - \frac{a}{q} + \frac{b}{q}, 1\}$. If the minimum is 1, then the value of the implication obviously belongs to $\mathcal{N}$. Otherwise, the value is $\frac{q-a+b}{q}$ which also belongs to $\mathcal{N}$: since $a, b \in [0, q]$ and $1 - \frac{a}{q} + \frac{b}{q} \leq 1$, it follows that $(1 - a + b) \in [0, q]$. □

Now, we will assume that $N^{fK} = \mathcal{N}$ and proceed similarly as in [9], which creates an optimized number of new elements (concepts, roles and axioms) with respect to previous approaches.

Without loss of generality, it can be assumed that $N^{fK} = \{\gamma_1, \ldots, \gamma_{|N^{fK}|}\}$ and $\gamma_i < \gamma_{i+1}, 1 \leq i \leq |N^{fK}| - 1$. It is easy to see that $\gamma_1 = 0$ and $\gamma_{|N^{fK}|} = 1$.

Let A^{fK} and R^{fK} be the set of atomic concepts and atomic roles occurring in a fuzzy KB $fK = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$. For each $\alpha, \beta \in N^{fK}$ with $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, for each $A \in A^{fK}$ and for each $R_A \in R^{fK}$, two new atomic concepts $A_{\geq \alpha}, A_{> \beta}$ and one new atomic role $R_{\geq \alpha}$ are introduced. $A_{\geq \alpha}$ represents the crisp set of individuals which are instance of A with degree higher or equal than α i.e the α-cut of A. The other new elements are defined in a similar way. The atomic elements $A_{>1}, A_{\geq 0}$ and $R_{\geq 0}$ are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB.

A concept of the form $A_{\geq \alpha}$ may be used for instance to represent an assertion of the form $\langle a \colon A \geq \alpha \rangle$, while $A_{> \alpha}$ can be used to represent an assertion of the form $\langle a \colon \neg A \geq 1 - \alpha \rangle$. Since role negation and axioms of the form $\tau > \beta$ are not allowed, elements of the form $R_{> \alpha}$ are not needed.

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each $1 \leq i \leq |N^{fK}| - 1, 2 \leq j \leq |N^{fK}| - 1$ and for each $A \in A^{fK}$, $T(N^{fK})$ is the smallest terminology containing the axioms: $A_{\geq \gamma_{i+1}} \sqsubseteq A_{> \gamma_i}, A_{> \gamma_j} \sqsubseteq A_{\geq \gamma_j}$. Similarly, for each atomic role $R \in R^{fK}$, we define $R(N^{fK})$ as the smallest terminology containing: $R_{\geq \gamma_{j+1}} \sqsubseteq R_{\geq \gamma_j}$.

4.2 Mapping Fuzzy Concepts, Roles and Axioms

Before showing how to represent the elements of the KB using these new elements, we will illustrate with an example.

Example 1. Consider a fuzzy assertion $\tau = \langle a : A_1 \sqcap A_2 \geq 0.5 \rangle$ and $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\}$. Every model $\mathcal{I}$ of τ satisfies $\max\{A_1^{\mathcal{I}}(a) + A_2^{\mathcal{I}}(a) - 1, 0\} \geq 0.5$. Hence, it follows that $A_1^{\mathcal{I}}(a) + A_2^{\mathcal{I}}(a) - 1 \geq 0.5 \Leftrightarrow A_1^{\mathcal{I}}(a) + A_2^{\mathcal{I}}(a) \geq 1.5$. Now, we do not know exactly the degrees of truth of $A_1^{\mathcal{I}}(a)$ and $A_2^{\mathcal{I}}(a)$, but we now that they belong to $\mathcal{N}$, so there are six possibilities:

$A_1^{\mathcal{I}}(a)$	$A_2^{\mathcal{I}}(a)$	$(A_1 \sqcap A_2)^{\mathcal{I}}(a)$
0.5	1	0.5
0.75	0.75	0.5
0.75	1	0.75
1	0.5	0.5
1	0.75	0.75
1	1	1

Hence, we can think of a crisp model satisfying $a : (A_{1 \geq 0.5} \sqcap A_{2 \geq 1}) \sqcup (A_{1 \geq 0.75} \sqcap A_{2 \geq 0.75}) \sqcup (A_{1 \geq 0.75} \sqcap A_{2 \geq 1}) \sqcup (A_{1 \geq 1} \sqcap A_{2 \geq 0.5}) \sqcup (A_{1 \geq 1} \sqcap A_{2 \geq 0.75}) \sqcup (A_{1 \geq 1} \sqcap A_{2 \geq 1})$.

But this (crisp) assertion can be optimized (see below Proposition 2) since it is satisfiable if the following is: $a : (A_{1 \geq 0.5} \sqcap A_{2 \geq 1}) \sqcup (A_{1 \geq 0.75} \sqcap A_{2 \geq 0.75}) \sqcup (A_{1 \geq 1} \sqcap A_{2 \geq 0.5})$. □

Proposition 2. *Let $A \sqsubseteq B_1 \sqcup B_2$ be a crisp GCI with B_1 subsuming B_2. $A \sqsubseteq B_1 \sqcup B_2$ is satisfiable if and only if $A \sqsubseteq B_1$ is.*

Proof. If $A \sqsubseteq B_1 \sqcup B_2$ is satisfiable then there exists a model $\mathcal{I}$ such that $A^{\mathcal{I}} \subseteq B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}}$. Since B_1 subsumes B_2, $B_2^{\mathcal{I}} \subseteq B_1^{\mathcal{I}}$, and hence $B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}} = B_1^{\mathcal{I}}$, so $A^{\mathcal{I}} \subseteq B_1^{\mathcal{I}}$ and hence $A \sqsubseteq B_1$ holds. The other direction is similar. □

We define $\mathcal{N}^+ = \{x \in \mathcal{N} : x \neq 0\}$. Concept and role expressions are reduced using mapping ρ, as shown in Table 1. Notice that expressions of the form $\rho(A, \geq 0)$ and $\rho(A, \leq 1)$ cannot appear, because there exist some restrictions on the degree of the axioms in the fuzzy KB. The same also holds for $\top$, $\bot$ and R_A. Moreover, expressions of the form $\rho(R, \lhd \gamma)$ cannot appear either.

Axioms are reduced as in Table 2, where σ maps fuzzy axioms into crisp assertions and κ maps fuzzy TBox (resp. RBox) axioms into crisp TBox (resp. RBox) axioms. We note $\sigma(\mathcal{A})$ (resp. $\kappa(fK, \mathcal{T})$, $\kappa(fK, \mathcal{R})$) the union of the reductions of every axiom in $\mathcal{A}$ (resp. $\mathcal{T}$, $\mathcal{R}$).

Table 1. Mapping of concept and role expressions

x	y	$\rho(x,y)$
$\top$	$\geq \alpha$	$\top$
$\top$	$\leq \beta$	$\bot$
$\bot$	$\geq \alpha$	$\bot$
$\bot$	$\leq \beta$	$\top$
A	$\geq \alpha$	$A_{\geq \alpha}$
A	$\leq \beta$	$\neg A_{> \gamma}$
R_A	$\geq \alpha$	$R_{\geq \alpha}$
$\neg C$	$\bowtie \gamma$	$\rho(C, \bowtie^{-} 1 - \gamma)$
$C \sqcap D$	$\geq \alpha$	$\bigsqcup_{\gamma_1, \gamma_2} \rho(C, \geq \gamma_1) \sqcap \rho(D, \geq \gamma_2)$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$, $\gamma_1 + \gamma_2 = 1 + \alpha$
$C \sqcap D$	$\leq \beta$	$\rho(\neg C, \geq 1 - \beta) \sqcup \rho(\neg D, \geq 1 - \beta)$
$C \sqcup D$	$\geq \alpha$	$\rho(C, \geq \alpha) \sqcup \rho(D, \geq \alpha) \bigsqcup_{\gamma_1, \gamma_2} \rho(C, \geq \gamma_1) \sqcap \rho(D, \geq \gamma_2)$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$, $\gamma_1 + \gamma_2 = \alpha$
$C \sqcup D$	$\leq \beta$	$\rho(\neg C, \geq 1 - \beta) \sqcap \rho(\neg D, \geq 1 - \beta)$
$\exists R.C$	$\geq \alpha$	$\bigsqcup_{\gamma_1, \gamma_2} \exists \rho(R, \geq \gamma_1).\rho(C, \geq \gamma_2)$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$, $\gamma_1 + \gamma_2 = 1 + \alpha$
$\exists R.C$	$\leq \beta$	$\rho(\forall R.\neg C, \geq 1 - \beta)$
$\forall R.C$	$\geq \alpha$	$\bigsqcap_{\gamma_1, \gamma_2} \forall \rho(R, \geq \gamma_1).\rho(C, \geq \gamma_2)$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$ and $\gamma_1 = \gamma_2 + 1 - \alpha$
$\forall R.C$	$\leq \beta$	$\rho(\exists R.\neg C, \geq 1 - \beta)$

Table 2. Reduction of the axioms

$\sigma(\langle a : C \geq \alpha \rangle)$	$a : \rho(C, \geq \alpha)$
$\sigma(\langle a : C \leq \beta \rangle)$	$a : \rho(C, \leq \beta)$
$\sigma(\langle (a,b) : R \geq \alpha \rangle)$	$(a,b) : \rho(R, \geq \alpha)$
$\kappa(\langle C \sqsubseteq D \geq \alpha \rangle)$	$\bigcup_{\gamma_1, \gamma_2} \{\rho(C, \geq \gamma_1) \sqsubseteq \rho(D, \geq \gamma_2)\}$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$ and $\gamma_1 = \gamma_2 + 1 - \alpha$
$\kappa(\langle R \sqsubseteq R' \geq \alpha \rangle)$	$\bigcup_{\gamma_1, \gamma_2} \{\rho(R, \geq \gamma_1) \sqsubseteq \rho(R', \geq \gamma_2)\}$ for every pair $\langle \gamma_1, \gamma_2 \rangle$ such that $\gamma_1, \gamma_2 \in \mathcal{N}^+$ and $\gamma_1 = \gamma_2 + 1 - \alpha$

4.3 Properties of the Reduction

Summing up, a fuzzy KB $fK = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ is reduced into a KB $\mathcal{K}(fK) = \langle \sigma(\mathcal{A}), T(N^{fK}) \cup \kappa(fK, \mathcal{T}),\ R(N^{fK}) \cup \kappa(fK, \mathcal{R}) \rangle$. The reduction is reasoning preserving since the following theorem shows:

Theorem 1. *Assuming a truth space* $\{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{(q-1)}{q}, 1\}$*, a* Ł-$\mathcal{ALCH}$ *fuzzy KB* fK *is satisfiable iff* $\mathcal{K}(fK)$ *is satisfiable.*

Example 2. Let us a consider a fuzzy KB $fK = \{\langle a : \forall R.(C \sqcap D) \geq 0.75 \rangle, \langle (a,b) : R \geq 0.75 \rangle, \langle b : \neg C \geq 0.75 \rangle\}$ and assume a set of degrees of truth $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\}$ ($q = 4$). Note that the TBox and the RBox are empty.

This fuzzy KB is clearly unsatisfiable. From the third assertion it follows that $C^{\mathcal{I}}(b^{\mathcal{I}}) \leq 0.25$, and it can be seen that this implies that $(C \sqcap D)^{\mathcal{I}}(b^{\mathcal{I}}) = \max\{C^{\mathcal{I}}(b^{\mathcal{I}}) + D^{\mathcal{I}}(b^{\mathcal{I}}) - 1, 0\} \leq 0.25$. But from the two former assertions it follows that every fuzzy interpretation $\mathcal{I}$ has to satisfy $(C \sqcap D)^{\mathcal{I}}(b^{\mathcal{I}}) \geq 0.5$, which is a contradiction.

Now, let us compute the crisp representation of fK. Firstly, we create some new crisp atomic concepts associated to the set of atomic fuzzy concepts $A^{fK} = \{C, D\}$ (i.e. $C_{>0}, C_{\geq 0.25}, C_{>0.25}, C_{\geq 0.5}, C_{>0.5}, C_{\geq 0.75}, C_{>0.75}, C_{\geq 1}, D_{>0}, D_{\geq 0.25}, D_{>0.25}, D_{\geq 0.5}, D_{>0.5}, D_{\geq 0.75}, D_{>0.75}, D_{\geq 1}$) and some new crisp atomic roles associated to the set of atomic fuzzy roles $R^{fK} = \{R\}$ (i.e. $R_{\geq 0.25}, R_{\geq 0.5}, R_{\geq 0.75}, R_{\geq 1}$).

Now we create some new axioms to preserve the semantics of these elements:

- $A^{fK} = \{C_{\geq 1} \sqsubseteq C_{>0.75}, C_{>0.75} \sqsubseteq C_{\geq 0.75}, C_{\geq 0.75} \sqsubseteq C_{>0.5}, C_{>0.5} \sqsubseteq C_{\geq 0.5}, C_{\geq 0.5} \sqsubseteq C_{>0.25}, C_{>0.25} \sqsubseteq C_{\geq 0.25}, C_{\geq 0.25} \sqsubseteq C_{>0}, D_{\geq 1} \sqsubseteq D_{>0.75}, D_{>0.75} \sqsubseteq D_{\geq 0.75}, D_{\geq 0.75} \sqsubseteq D_{>0.5}, D_{>0.5} \sqsubseteq D_{\geq 0.5}, D_{\geq 0.5} \sqsubseteq D_{>0.25}, D_{>0.25} \sqsubseteq D_{\geq 0.25}, D_{\geq 0.25} \sqsubseteq D_{>0}\}$,
- $R^{fK} = \{R_{\geq 1} \sqsubseteq R_{\geq 0.75}, R_{\geq 0.75} \sqsubseteq R_{\geq 0.5}, R_{\geq 0.5} \sqsubseteq R_{\geq 0.25}\}$.

Now we are ready to compute $\sigma(\mathcal{A})$, including the reduction of the three fuzzy assertions in the fuzzy KB, that is:

- $\sigma(\langle (a,b) : R \geq 0.75 \rangle) = (a,b) : \rho(R, \geq 0.75) = (a,b) : R_{\geq 0.75}$.
- $\sigma(\langle b : \neg C \geq 0.75 \rangle) = b : \rho(\neg C, \geq 0.75) = b : \neg C_{>0.25}$.
- $\sigma(\langle a : \forall R.(C \sqcap D) \geq 0.75 \rangle) = a : \rho(\forall R.(C \sqcap D), \geq 0.75) = a : [\forall \rho(R, \geq 0.5).\rho(C \sqcap D, \geq 0.25) \sqcap \forall \rho(R, \geq 0.75).\rho(C \sqcap D, \geq 0.5) \sqcap \forall \rho(R, \geq 1).\rho(C \sqcap D, \geq 0.75)]$, where:
  - $\rho(R, \geq 0.5) = R_{\geq 0.5}$,
  - $\rho(C \sqcap D, \geq 0.25) = (C_{\geq 0.25} \sqcap D_{\geq 1}) \sqcup (C_{\geq 0.5} \sqcap D_{\geq 0.75}) \sqcup (C_{\geq 0.75} \sqcap D_{\geq 0.5}) \sqcup (C_{\geq 1} \sqcap D_{\geq 0.25})$,
  - $\rho(R, \geq 0.75) = R_{\geq 0.75}$,
  - $\rho(C \sqcap D, \geq 0.5) = (C_{\geq 0.5} \sqcap D_{\geq 1}) \sqcup (C_{\geq 0.75} \sqcap D_{\geq 0.75}) \sqcup (C_{\geq 1} \sqcap D_{\geq 0.5})$,
  - $\rho(R, \geq 1) = R_{\geq 1}$,
  - $\rho(C \sqcap D, \geq 0.75) = (C_{\geq 0.75} \sqcap D_{\geq 1}) \sqcup (C_{\geq 1} \sqcap D_{\geq 0.75})$,

It can be seen that the (crisp) KB $\mathcal{K}(fK) = \langle \sigma(\mathcal{A}), T(N^{fK}), R(N^{fK}) \rangle$ is unsatisfiable. □

Regarding the complexity, the size of the resulting KB is $\mathcal{O}(n^k)$, where k is the maximal depth of the concepts appearing in the fuzzy KB. The depth of a concept is inductively defined as follows:

- $depth(A) = depth(\neg A) = depth(\top) = depth(\bot) = 1$,
- $depth(\forall R.C) = depth(\exists R.C) = 1 + depth(C)$,
- $depth(C \sqcap D) = depth(C \sqcup D) = 1 + max\{depth(C), depth(D)\}$.

We recall that under Zadeh semantics, the size of the resulting KB is quadratic. In our case we need to generate more and more complex axioms, because we cannot infer the exact values of the elements which take part of a complex concept, so we need to build disjunctions or conjunctions over all possible degrees of truth.

Finally, our reduction procedure is modular and it could be applied to more expressive DLs. In particular, adding fuzzy nominals (indicated with the letter $\mathcal{O}$) and inverse roles (indicated with the letter $\mathcal{I}$) is straightforward because their semantics do not depend on any particular choice of fuzzy operators, so they can be dealt with in the same way as for the Zadeh family [7].

Fuzzy nominals allow to represent extensive definitions of fuzzy sets. Let o_i denote a named individual and let $\alpha_i \in (0, 1]$, for $i \in \{1, \dots, m\}$. The syntax for fuzzy nominal concepts is $\{\alpha_1/o_1, \dots, \alpha_m/o_m\}$ and their semantics is:

$$\{\alpha_1/o_1, \dots, \alpha_m/o_m\}^{\mathcal{I}}(a) = \sup_{i \mid a \in \{o_i^{\mathcal{I}}\}} \alpha_i$$

The syntax for inverse roles is the same as in the crisp case, i.e. R^- denotes the inverse of the role R, and their semantics is:

$$(R^-)^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(b, a)$$

As anticipated, mapping ρ can be extended in order to deal with these constructors in the following way:

x	y	$\rho(x, y)$
$\{\alpha_1/o_1, \dots, \alpha_m/o_m\}$	$\bowtie \gamma$	$\{o_i \mid \alpha_i \bowtie \gamma, 1 \leq i \leq n\}$
R^-	$\rhd \alpha$	$\rho(R, \rhd \alpha)^-$

Hence Ł-$\mathcal{ALCHOI}$ can be mapped into crisp $\mathcal{ALCHOI}$.

However, we point out that how to represent transitive roles and cardinality restrictions, which are the constructs which remain to reach a fuzzy extension of $\mathcal{SHOIN}$ (and hence and eventually OWL DL), remains an open issue.

5 Conclusions and Future Work

In this paper we have shown how to reduce a fuzzy extension of $\mathcal{ALCHOI}$ under Łukasiewicz semantics, assuming a fixed set of allowed degrees of truth, into $\mathcal{ALCHOI}$. This work means an important step towards the possibility of dealing with imprecise and vague knowledge in DLs, since it relies on existing languages and tools. Our work is more general than previous approaches which provide crisp representations of fuzzy DLs under Zadeh semantics ([4,5,7,8,9]). However, from a practical point of view, the size of the resulting KB is much more complex in this case, so the practical feasibility of this approach has to be empirically verified. The idea behind our reduction is modular and could be applied to more expressive DLs. In future work we plan to extend the expressiveness of the logic and to implement the proposed reduction, studying if it can be optimized in some particular common situations.

References

1. Baader, F., Calvanese, D., McGuinness, D.L., Nardi, D., Patel-Schneider, P.F.: The description logic handbook: Theory, implementation, and applications. Cambridge University Press, Cambridge (2003)
2. McGuinness, D.L., van Harmelen, F.: OWL Web Ontology Language overview (2004), `http://www.w3.org/TR/owl-features`
3. Lukasiewicz, T., Straccia, U.: An overview of uncertainty and vagueness in description logics for the semantic web. Technical Report INFSYS RR-1843-06-07, Institut für Informationssysteme, Technische Universität Wien (2006)
4. Straccia, U.: Transforming fuzzy description logics into classical description logics. In: Alferes, J.J., Leite, J.A. (eds.) JELIA 2004. LNCS (LNAI), vol. 3229, pp. 385–399. Springer, Heidelberg (2004)
5. Straccia, U.: Description logics over lattices. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 14(1), 1–16 (2006)
6. Hájek, P.: Making fuzzy description logics more general. Fuzzy Sets and Systems 154(1), 1–15 (2005)
7. Bobillo, F., Delgado, M., Gómez-Romero, J.: A crisp representation for fuzzy SHOIN with fuzzy nominals and general concept inclusions. In: Proceedings of the 2nd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2006), CEUR Workshop Proceedings, vol. 218 (2006)
8. Stoilos, G., Stamou, G.: Extending fuzzy description logics for the semantic web. In: Proceedings of the 3rd International Workshop on OWL: Experiences and Directions (OWLED 2007) (2007)
9. Bobillo, F., Delgado, M., Gómez-Romero, J.: Optimizing the crisp representation of the fuzzy description logic SROIQ. In: Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007) (to appear, 2007)
10. Zadeh, L.A.: Fuzzy sets. Information and Control 8, 338–353 (1965)
11. Klir, G.J., Yuan, B.: Fuzzy sets and fuzzy logic: Theory and applications. Prentice-Hall, Englewood Cliffs (1995)
12. Straccia, U.: A fuzzy description logic for the semantic web. In: Fuzzy Logic and the Semantic Web. Capturing Intelligence, vol. 1, pp. 73–90. Elsevier Science, Amsterdam (2006)
13. Straccia, U.: Reasoning within fuzzy description logics. Journal of Artificial Intelligence Research 14, 137–166 (2001)