The Approximate Well-founded Semantics for Logic Programs with Uncertainty

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A Proofs

Theorem 1. *Consider $\mathcal{T} \times \mathcal{T}$ with the orderings $\preceq_t$ and $\preceq_k$. Then*

1. *$\otimes^t, \oplus^t, \otimes^k, \oplus^k$ and the extensions of combination functions are continuous (and, thus, monotonic) w.r.t. $\preceq_t$ and $\preceq_k$;*
2. *any extended negation function is monotonic w.r.t. $\preceq_k$;*
3. *if the negation function satisfies the de Morgan laws, i.e. $\forall a, b \in \mathcal{T}. \neg(a \oplus b) = \neg a \otimes \neg b$ then the extended negation function is continuous w.r.t. $\preceq_k$.*

Proof. We proof only the last item, as the others are immediate. Consider a chain of intervals $x_0 \preceq_k x_1 \preceq_k \ldots$, where $x_j = [a_j; b_j]$ with $a_j, b_j \in \mathcal{T}$. To show the continuity of the extended negation function w.r.t. $\preceq_k$, we show that $\neg \oplus^k_{j \geq 0} x_j = \oplus^k_{j \geq 0} \neg x_j$. Indeed, the following holds:

$$\begin{aligned} \neg \oplus^k_{j \geq 0} x_j &= \neg[\oplus_{j \geq 0} a_j; \otimes_{j \geq 0} b_j] = [\neg \otimes_{j \geq 0} b_j; \neg \oplus_{j \geq 0} a_j] = [\oplus_{j \geq 0} \neg b_j; \otimes_{j \geq 0} \neg a_j] \\ &= \oplus_{j \geq 0}[\neg b_j; \neg a_j] \quad = \quad \oplus_{j \geq 0} \neg [a_j; b_j] \quad = \oplus^k_{j \geq 0} \neg x_j \end{aligned}$$

Theorem 2. *For any np-program P, T_P is monotonic and, if the de Morgan laws hold, continuous w.r.t. $\preceq_k$.*

Proof. The proof of monotonicity is easy. To proof the continuity w.r.t. $\preceq_k$, consider a chain of interpretations $I_0 \preceq_k I_1 \preceq_k \ldots$. To proof continuity of T_P, we show that for any $A \in \mathcal{B}_P$,

$$T_P(\oplus^k_{j \geq 0} I_j)(A) = \oplus^k_{j \geq 0} T_P(I_j)(A) \tag{1}$$

As $\mathcal{C}_P$ is a complete lattice, the sequence $I_0 \preceq_k I_1 \preceq_k \ldots$ has a least upper bound, say $\bar{I} = \oplus^k_{j \geq 0} I_j$. For any $B \in \mathcal{B}_P$, we have $\oplus^k_{j \geq 0} I_j(B) = \bar{I}(B)$ and, from Theorem 2, $\oplus^k_{j \geq 0} I_j(\neg B) = \oplus^k_{j \geq 0} \neg I_j(B) = \neg \oplus^k_{j \geq 0} I_j(B) = \neg \bar{I}(B)$ and, thus, for any literal or certainty value L,

$$\oplus^k_{j \geq 0} I_j(L) = \bar{I}(L) \tag{2}$$

Now, consider the finite set (P^* is finite) of all ground rules $r_1, \ldots, r_k$ having A as head, where $r_i = A \overset{\alpha_i}{\leftarrow} L^i_1, ..., L^i_{n_i}; \langle f_d, f^i_p, f^i_c \rangle)$. Let us evaluate the left hand side of Equation 1.

$$T_P(\oplus^k_{j \geq 0} I_j)(A) = T_P(\bar{I})(A) = f_d(\{ f^i_p([\alpha_i; \alpha_i], f^i_c(\{ \bar{I}(L^i_1), \ldots, \bar{I}(L^i_{n_i} \})) \colon 0 \leq i \leq k \})$$

On the other hand side,

$$\oplus^k_{j \geq 0} T_P(I_j)(A) = \oplus^k_{j \geq 0} f_d(\{ f^i_p([\alpha_i; \alpha_i], f^i_c(\{ I_j(L^i_1), \ldots, I_j(L^i_{n_i} \})) \colon 0 \leq i \leq k \})$$

But, f_d, f^i_p and f^i_c are continuous and, thus, by Equation 2, the following holds.

$$\begin{aligned} \oplus^k_{j \geq 0} T_P(I_j)(A) &= f_d(\{ \oplus^k_{j \geq 0} \{ f^i_p([\alpha_i; \alpha_i], f^i_c(\{ I_j(L^i_1), \ldots, I_j(L^i_{n_i} \})) \colon 0 \leq i \leq k \} \}) \\ &= f_d(\{ f^i_p([\alpha_i; \alpha_i], \oplus^k_{j \geq 0} \{ f^i_c(\{ I_j(L^i_1), \ldots, I_j(L^i_{n_i} \}) \}) \colon 0 \leq i \leq k \}) \\ &= f_d(\{ f^i_p([\alpha_i; \alpha_i], f^i_c(\{ \oplus^k_{j \geq 0} I_j(L^i_1), \ldots, \oplus^k_{j \geq 0} I_j(L^i_{n_i} \})) \colon 0 \leq i \leq k \}) \\ &= f_d(\{ f^i_p([\alpha_i; \alpha_i], f^i_c(\{ \bar{I}(L^i_1), \ldots, \bar{I}(L^i_{n_i} \})) \colon 0 \leq i \leq k \}) \end{aligned}$$

Therefore, Equation 1 holds and, thus, T_P is continuous.