Syntax and Semantics of fuzzyDL

Fernando Bobillo and Umberto Straccia

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1. Comments Any line beginning with # or % is considered a comment.

2. Fuzzy operators. $\ominus, \oplus, \ominus$ and $\Rightarrow$ denote a t-norm, t-conorm, negation function and implication function respectively; $\alpha, \beta \in [0, 1]$.

Łukasiewicz negation	$\ominus_{Ł}\alpha$	$1 - \alpha$
Gödel t-norm	$\alpha \otimes_G \beta$	$\min\{\alpha, \beta\}$
Łukasiewicz t-norm	$\alpha \otimes_{Ł} \beta$	$\max\{\alpha + \beta - 1, 0\}$
Gödel t-conorm	$\alpha \oplus_G \beta$	$\max\{\alpha, \beta\}$
Łukasiewicz t-conorm	$\alpha \oplus_{Ł} \beta$	$\min\{\alpha + \beta, 1\}$
Gödel implication	$\alpha \Rightarrow_G \beta$	$\begin{cases} 1, & \text{if } \alpha \leq \beta \\ \beta, & \text{if } \alpha > \beta \end{cases}$
Łukasiewicz implication	$\alpha \Rightarrow_{Ł} \beta$	$\min\{1, 1 - \alpha + \beta\}$
Kleene-Dienes implication	$\alpha \Rightarrow_{KD} \beta$	$\max\{1 - \alpha, \beta\}$
Zadeh'set inclusion	$\alpha \Rightarrow_Z \beta$	1 iff $\alpha \leq \beta$, 0 otherwise

The reasoner can accept three different semantics, which are used to interpret $\ominus, \oplus, \ominus$ and $\Rightarrow$.

- Zadeh semantics: Łukasiewicz negation, Gödel t-norm, Gödel t-conorm and Kleene-Dienes implication (except in GCIs, where we have that the degree of membership to the subsumed concept should be less or equal than the degree of membership to the subsumer concept). This semantics is included for compatibility with earlier papers about fuzzy description logics.
- Łukasiewicz semantics: Łukasiewicz negation, Łukasiewicz t-norm, Łukasiewicz t-conorm and Łukasiewicz implication.
- Classical semantics: classical (crisp) conjunction, disjunction, negation and implication.

Syntax to define the semantics of the knowledge base:

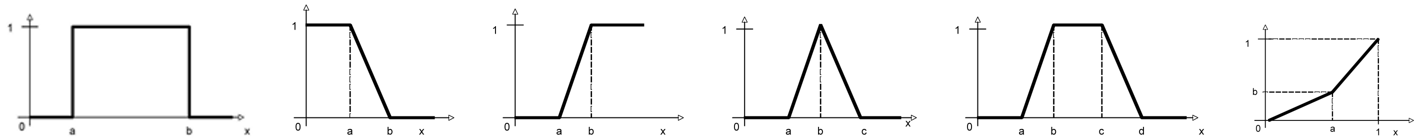
(define-fuzzy-logic [lukasiewicz — zadeh — classical])

3. Truth constants. Truth constants can be defined as follows (and later on, they can be used as the lower bound of a fuzzy axiom): (define-truth-constant constant n), where n is a rational number in $[0, 1]$.

4. Concept modifiers. Modifiers change the membership function of a fuzzy concept.

(define-modifier CM linear-modifier(c))	linear hedge with $c > 0$ (Figure 1 (f))
(define-modifier CM triangular-modifier(a,b,c))	triangular function (Figure 1 (d))

5. Concrete Fuzzy Concepts. Concrete Fuzzy Concepts (CFCs) define a name for a fuzzy set with an explicit fuzzy membership function (we assume $a \leq b \leq c \leq d$).



(a) (b) (c) (d) (e) (f)

Figure 1: (a) Crisp value; (b) L-function; (c) R-function; (d) (b) Triangular function; (e) Trapezoidal function; (f) Linear hedge

(define-fuzzy-concept CFC crisp(k1,k2,a,b))	crisp interval (Figure 1 (a))
(define-fuzzy-concept CFC left-shoulder(k1,k2,a,b))	left-shoulder function (Figure 1 (b))
(define-fuzzy-concept CFC right-shoulder(k1,k2,a,b))	right-shoulder function (Figure 1 (c))
(define-fuzzy-number CFC triangular(k1,k2,a,b,c))	triangular function (Figure 1 (d))
(define-fuzzy-concept CFC trapezoidal(k1,k2,a,b,c,d))	trapezoidal function (Figure 1 (e))
(define-fuzzy-concept CFC linear(k1,k2,a,b))	linear function (Figure 1 (f))
(define-fuzzy-concept CFC modified(mod,F))	modified datatype

6. Fuzzy Numbers. Firstly, if fuzzy numbers are used, one has to define the range $[k_1, k_2] \subseteq \mathbb{R}$ as follows:

```
(define-fuzzy-number-range k1 k2)
```

Let `fi` be a fuzzy number (a_i, b_i, c_i) $(a \leq b \leq c)$, and $n \in \mathbb{R}$. Valid fuzzy number expressions (see Figure 1 (d)) are:

name	fuzzy number definition	*name*
(a, b, c)	fuzzy number	(a, b, c)
n	real number	(n, n, n)
(f+ f1 f2 … fn)	addition	$(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i)$
(f- f1 f2)	substraction	$(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(f* f1 f2 … fn)	product	$(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i, \prod_{i=1}^{n} c_i)$
(f/ f1 f2)	division	$(a_1/c_2, b_1/b_2, c_1/a_2)$

Fuzzy numbers can be named as:

```
(define-fuzzy-number name fuzzyNumberExpression)
```

7. Features. Features are functional datatype attributes.

(functional F)	Firstly, the feature is defined. Then we set the range
(range F *integer* k1 k2)	The range are integer numbers in $[k_1, k_2]$
(range F *real* k1 k2)	The range are rational number in $[k_1, k_2]$
(range F *string*)	The range are strings
(range F *boolean*)	The range are booleans

8. Datatype restrictions.

(some F [$\geq var$])	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \geq var)]$
(some F [$\geq f(F_1, \ldots, F_n)$])	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \geq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(some F [$\geq FN$])	at least datatype restriction	$\sup_{b,b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \geq b') \otimes FN^{\mathcal{I}}(b')]$
(some F [$\leq var$])	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \leq var)]$
(some F [$\leq f(F_1, \ldots, F_n)$])	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \leq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(some F [$\leq FN$])	at most datatype restriction	$\sup_{b,b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b \leq b') \otimes FN^{\mathcal{I}}(b')]$
(some F { var })	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b = var)]$
(some F { $f(F_1, \ldots, F_n)$})	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes (b = f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(some F { FN })	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x,b) \otimes FN^{\mathcal{I}}(b)]$

In datatype restrictions, the variable *var* has to be declared (free var) before its use in a datatype restriction, as defined in paragraph 9. below. The value for b has to be in the range $[k1, k2] \subseteq [-k_\infty, k_\infty]$ of the feature F, and the values for var, $f(F_1, \ldots, F_n)$ and the range of FN have to be in $[-k_\infty, k_\infty]$, where k_∞ is the maximal representable integer, currently $2 \cdot 10^{12}$. Note also that in datatype restrictions, the variable *var* may be replaced with a value, i.e., an integer, a real, a string, or a boolean constant (true, false) depending on the range of the feature F, although in the case of booleans, <= and >= are not allowed.

Furthermore, f is defined as follows:

$$
\begin{array}{ll}
f(F_1, \ldots, F_n) \rightarrow & F \\
 & real \\
 & (nF) \mid (n * F) \\
 & (F_1 - F_2) \\
 & (F_1 + F_2 + \ldots + F_n)
\end{array}
$$

9. Constraints. Constraints are of the form (constraints $\langle$ constraint-i $\rangle$ +), where $\langle$ constraint-i $\rangle$ is one of the following (with $OP = \geq \mid \leq \mid =$.):

(a1 * var1 + ... + ak * vark OP number)	linear inequation	$a_1 var_1 + \ldots + a_k * var_k \; OP \; number$
(binary var)	binary variable	$var \in \{0, 1\}$
(free var)	binary variable	$var \in (-\infty, \infty)$

10. Show statements.

(show-concrete-fillers F1 ... Fn)	show value of the fillers of $F_1 \ldots F_n$
(show-concrete-fillers-for a F1 ... Fn)	show value of the fillers of $F_1 \ldots F_n$ for a
(show-concrete-instance-for a F C1 ... Cn)	show degrees of being the F filler of a an instance of C_i
(show-abstract-fillers R1 ... Rn)	show fillers of $R_1 \ldots R_n$ and membership to any concept
(show-abstract-fillers-for a R1 ... Rn)	show fillers of $R_1 \ldots R_n$ for a and membership to any concept
(show-concepts a1 ... an)	show membership of $a_1 \ldots a_n$ to any concept
(show-instances C1 ... Cn)	show value of the instances of the concepts $C_1 \ldots C_n$
(show-variables x1 ... xn)	show value of the variables $x_1 \ldots x_n$
(show-language)	show language of the KB, from $\mathcal{ALC}$ to $\mathcal{SHIF}(D)$

where C_i is the name of a defined concrete fuzzy concept. We assume that an abstract role R appears in at most one statement of the forms show-abstract-fillers? or show-abstract-fillers-for?.

11. Crisp declarations.

(crisp-concept C1 ... Cn)	concepts $C_1 \ldots C_n$ are crisp
(crisp-role R1 ... Rn)	roles $R_1 \ldots R_n$ are crisp

12. Concept expressions.

top	top concept	1
bottom*	bottom concept	0
A	atomic concept	$A^{\mathcal{I}}(x)$
(and C1 C2)	concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes {C_2}^{\mathcal{I}}(x)$
(g-and C1 C2)	Gödel conjunction	${C_1}^{\mathcal{I}}(x) \otimes_G {C_2}^{\mathcal{I}}(x)$
(l-and C1 C2)	Łukasiewicz conjunction	${C_1}^{\mathcal{I}}(x) \otimes_{\text{Ł}} {C_2}^{\mathcal{I}}(x)$
(or C1 C2)	concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus {C_2}^{\mathcal{I}}(x)$
(g-or C1 C2)	Gödel disjunction	${C_1}^{\mathcal{I}}(x) \oplus_G {C_2}^{\mathcal{I}}(x)$
(l-or C1 C2)	Łukasiewicz disjunction	${C_1}^{\mathcal{I}}(x) \oplus_{\text{Ł}} {C_2}^{\mathcal{I}}(x)$
(not C1)	concept negation	$\ominus_{\text{Ł}} {C_1}^{\mathcal{I}}(x)$
(implies C1 C2)	concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x)$
(g-implies C1 C2)	Gödel implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x)$
(l-implies C1 C2)	Łukasiewicz implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}} {C_2}^{\mathcal{I}}(x)$
(kd-implies C1 C2)	Kleene-Dienes implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x)$
(all R C1)	universal role restriction	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \Rightarrow {C_1}^{\mathcal{I}}(y)$
(some R C1)	existential role restriction	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x,y) \otimes {C_1}^{\mathcal{I}}(y)$
(some R a)	individual value restriction	$R^{\mathcal{I}}(x,a)$
(ua s C1)	upper approximation	$\sup_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$
(la s C1)	lower approximation	$\inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$
(tua s C1)	tight upper approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x,z) \Rightarrow \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y)\}\}$
(lua s C1)	loose upper approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x,z) \otimes \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \otimes C^{\mathcal{I}}(y)\}\}$
(tla s C1)	tight lower approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x,z) \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(lla s C1)	loose lower approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x,z) \otimes \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y,z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(self S)	local reflexivity concept	$S^{\mathcal{I}}(x)(x,x)$
(CM C1)	modifier applied to concept	$f_{\mathfrak{m}}({C_1}^{\mathcal{I}}(x))$
(CFC)	concrete fuzzy concept	$CFC^{\mathcal{I}}(x)$
(FN)	fuzzy number	$FN^{\mathcal{I}}(x)$
([>= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \geq w \\ 0, & \text{otherwise} \end{cases}$
([<= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \leq w \\ 0, & \text{otherwise} \end{cases}$
(n C1)	weighted concept	$n{C_1}^{\mathcal{I}}(x)$
(w-sum (n1 C1) ... (nk Ck))	weighted sum	$n_1{C_1}^{\mathcal{I}}(x) + \ldots + n_k{C_k}^{\mathcal{I}}(x)$
(w-max (v1 C1) ... (vk Ck))	weighted maximum	$\max_{i=1}^{k} \min\{v_i, x_i\}$
(w-min (v1 C1) ... (vk Ck))	weighted minimum	$\min_{i=1}^{k} \max\{1 - v_i, x_i\}$
(w-sum-zero (n1 C1) ... (nk Ck))	weighted sum-zero	$\begin{cases} 0 & \text{if } {C_i}^{\mathcal{I}}(x) = 0 \text{ for some } i \in \{1, \ldots, n\} \\ \text{w-sum} & \text{otherwise} \end{cases}$
(owa (w1 ... wn) (C1 ... Cn))	OWA aggregation operator	$\sum_{i=1}^{n} w_i y_i$
(q-owa q C1 ... Cn)	quantifier-guided OWA	Same as OWA taking $w_i = q(i/n) - q((i-1)/n)$
(choquet (w1 ... wn) (C1 ... Cn))	Choquet integral	$y_1 \cdot w_1 + \sum_{i=2}^{n} (y_i - y_{i-1}) w_i$
(sugeno (v1 ... vn) (C1 ... Cn))	Sugeno integral	$\max_{i=1}^{n} \{\min\{y_i, mu_i\}\}$
(q-sugeno (v1 ... vn) (C1 ... Cn))	Quasi-Sugeno integral	$\max_{i=1}^{n} \{y_i \otimes_{\text{Ł}} mu_i\}$
(DR)	datatype restriction	$DR^{\mathcal{I}}(x)$

where:

- a is an individual
- $n_1, \ldots, n_k \in [0,1]$ with $\sum_{i=1}^{k} n_i \leq 1$,
- $w_1, \ldots, w_k \in [0,1]$ with $\sum_{i=1}^{k} w_i = 1$,
- $v_1, \ldots, v_k \in [0,1]$ with $\max_{i=1}^{k} k_i = 1$,
- q is a quantifier (defined as a right-shoulder or a linear function),

- w is a variable or a real number in $[0, 1]$,
- y_i is the i-largest of the ${C_i}^{\mathcal{I}}(x)$,
- mu_i is defined as follows: $mu_1 = ow_1$, $mu_i = ow_i \oplus mu_{i-1}$ for $i \in \{2, \ldots, n\}$,
- ow_i is the weight v_i of the i-largest of the ${C_i}^{\mathcal{I}}(x)$.
- Fuzzy numbers can only appear in existential, universal and datatype restrictions.
- In threshold concepts var may be replaced with w.
- Fuzzy relations s should be previously defined as fuzzy similarity relation or a fuzzy equivalence relation as `(define-fuzzy-similarity s)` or `(define-fuzzy-equivalence s)`, respectively.

Important note: The reasoner restricts the calculus to witnessed models.

13. Axioms.

(instance a C1 [d])	concept assertion	${C_1}^{\mathcal{I}}(a^{\mathcal{I}}) \geq d$
(related a b R [d])	role assertion	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq d$
(implies C1 C2 [dN])	GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x) \geq dN$
(g-implies C1 C2 [dN])	Gödel GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x) \geq dN$
(kd-implies C1 C2 [dN])	Kleene-Dienes GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x) \geq dN$
(l-implies C1 C2 [dN])	Łukasiewicz GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x) \geq dN$
(z-implies C1 C2 [dN])	Zadeh's set inclusion GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_Z {C_2}^{\mathcal{I}}(x) \geq dN$
(define-concept A C)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) = C^{\mathcal{I}}(x)$
(define-primitive-concept A C)	concept subsumption	$\inf_{x \in \Delta^{\mathcal{I}}} A^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$
(equivalent-concepts C1 C2)	concept definition	$\forall_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) = {C_2}^{\mathcal{I}}(x)$
(disjoint C1 ... Ck)	concept disjointness	$\forall_{i,j \in \{1,\ldots,k\}, i<j}$ (implies (g-and Ci Cj) *bottom*)
(disjoint-union C1 ... Ck)	disjoint union	(disjoint C2 ... Ck) and C1 = (or C2 ... Ck)
(range R C1)	range restriction	(implies *top* (all RN C))
(domain R C1)	domain restriction	(implies (some RN *top*) C)
(functional R)	functional role	$R^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(a, c) \rightarrow b = c$
(inverse-functional R)	inverse functional role	$R^{\mathcal{I}}(b, a) = R^{\mathcal{I}}(c, a) \rightarrow b = c$
(reflexive R)	reflexive role	$\forall a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, a) = 1.$
(symmetric R)	symmetric role	$\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(b, a).$
(transitive R)	transitive role	$\forall_{a,b \in \Delta^{\mathcal{I}}}\ R^{\mathcal{I}}(a, b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a, c) \otimes R^{\mathcal{I}}(c, b).$
(implies-role R1 R2 [dN])	RIA	$\inf_{x,y \in \Delta^{\mathcal{I}}} {R_1}^{\mathcal{I}}(x, y) \Rightarrow_{Ł} {R_2}^{\mathcal{I}}(x, y) \geq dN$
(inverse R1 R2)	inverse role	${R_1}^{\mathcal{I}} \equiv ({R_2}^{\mathcal{I}})^{-}$

where dN is a degree of truth defined by means of a rational number in $[0, 1]$ and d is a degree of truth that can be: *(i)* a variable, *(ii)* an already defined truth constant, *(iii)* a rational number in $[0, 1]$, *(iv)* a linear expression.

Notes: Transitive roles cannot be functional. In Zadeh logic, $\Rightarrow$ is Zadeh's set inclusion.

14. Queries.

(sat?)	Is $\mathcal{K}$ consistent?
(max-instance? a C)	$\sup\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(min-instance? a C)	$\inf\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(all-instances? C)	(min-instance? a C) for every individual of $\mathcal{K}$
(max-related? a b R)	$\sup\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(min-related? a b R)	$\inf\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(max-g-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(min-g-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(max-l-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(min-l-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(max-kd-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(min-kd-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(max-sat? C [a])	$\sup_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(min-sat? C [a])	$\inf_{\mathcal{I}} \inf_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(max-var? var)	$\sup\{$var $\mid \mathcal{K}$ is consistent$\}$
(min-var? var)	$\inf\{$var $\mid \mathcal{K}$ is consistent$\}$
(defuzzify-lom? C_m a F)	Defuzzify the value of F using largest of the maxima
(defuzzify-mom? C_m a F)	Defuzzify the value of F using middle of the maxima
(defuzzify-som? C_m a F)	Defuzzify the value of F using smallest of the maxima
(bnp? f)	Computes the Best Non-Fuzzy Performance (BNP) of fuzzy number f

where concept C_m represents several Mamdani/Rules IF-THEN fuzzy rules expressing how to obtain the value of concrete feature F.

15. Computational Issues. In this short paragraph we advice the fuzzyDL user on various computational issues that may arise when using the reasoner.

fuzzyDL reasoning algorithms are based on a mixture of a fuzzy tableau algorithm, which transforms a reasoning problem into a Mixed Integer Linear Programming (MILP) problem which is solved by an operational research component (currently, Gurobi[1]).We refer the interested reader to [6].

A first computational issue may arise by the numerical solution identified by the underlying MILP solver. In some cases, the numerical precision adopted by the underlying MILP solver may not be enough to guarantee that the identified solution is indeed correct. This may happen for instance, in case very large, or very small (Java) numerical values are represented in the Knowledge Base, or values very close to 0, or in case the reasoner has to distinguish between numbers that are very close each other, although the precision of the MILP solver is not enough to distinguish them.

Another, orthogonal, issue is due to the fact that in some cases the reasoning problems addressed by fuzzyDL are undecidable (see, e.g. [1, 2, 3, 5, 6]). Currently, in such cases (see Remark 1 below) the correct behaviour of fuzzyDL is not guaranteed. For instance, for the KB [2, Example 2.3], fuzzyDL claims that the KB is satisfiable while this is not the case. Roughly, the problem arises in case a KB forces to generate infinitely many individuals with infinitely many truth degrees (see axioms (1)-(4) in [2, Example 2.3]). We didn't find any real-world ontology that intentionally exhibits such a behaviour so far.

In the following, let us introduce now three auxiliary definitions and then we will recap some of such cases for the interested reader. Let $L_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ for some natural number $n > 1$.

Definition 1 *The logic $\mathcal{ELC}$ is restricted to: (a) the concepts* *top*, and, some, not, *and* *bottom*, *and (b) the axioms concept assertion, role assertion, and GCI.*

[1] http://www.gurobi.com

Definition 2 *The $\mathcal{AL}$ logic is restricted to: (a) the concepts* *top*, and, some, *and* all, *and (b) the axioms concept assertion, role assertion, and GCI.*

Definition 3 *A TBox $\mathcal{T}$ is equivalent to the union of two disjoint sets of axioms $\mathcal{T}_g$ and $\mathcal{T}_u$ verifying the following conditions:*

1. *$\mathcal{T}_g$ is a set of GCIs of the form (implies *top* C α),*

2. *$\mathcal{T}_u = T_{def} \cup \mathcal{T}_{inc} \cup \mathcal{T}_{dom} \cup \mathcal{T}_{rg} \cup \mathcal{T}_{disj} \cup \mathcal{T}_{syn}$ is the disjoint union of*

   (a) *$\mathcal{T}_{def}$, which contains concept definitions only;*
   (b) *$\mathcal{T}_{inc}$, which contains GCIs of the form (implies A C [d]) only;*
   (c) *$\mathcal{T}_{dom}$, which contains domain restrictions only;*
   (d) *$\mathcal{T}_{rg}$, which contains range restrictions only;*
   (e) *$\mathcal{T}_{disj}$, which contains concept disjointness axioms only;*
   (f) *$\mathcal{T}_{syn}$, which contains axioms synonym of the form (define-concept A1 A2) only*

3. *there cannot be a concept name A that is a head of axioms in $\mathcal{T}_{def}$ and $\mathcal{T}_{inc}$; and*

4. *there cannot be (disjoint A B) $\in T_{disj}$, where both A, B are head of axioms in $\mathcal{T}_{def}$.*

If $\mathcal{T}_g = \emptyset$, the TBox $\mathcal{T}$ is called lazy unfoldable.

Now, we are ready to mention some undecidability results:

Remark 1 *For the following fuzzy DLs, the KB satisfiability problem is undecidable over* $[0, 1]$*:*

1. *$\mathcal{ELC}$ under Łukasiewicz logic. This result holds even if the GCIs are restricted to the case where the degree is* 1, *i.e., (implies C D 1).*

2. *$\mathcal{AL}$ with concept implication operator $\rightarrow$ and concept assertions of the form $\langle \alpha = n \rangle$ under Łukasiewicz logic.*

3. *$\mathcal{ELC}$ under SFL (Standard Fuzzy Logic, i.e. Zadeh semantics) with aggregation operators (weighted sum, OWAs, or fuzzy integrals). For example, weighted sum can encode Łukasiewicz t-conorm.*

4. *$\mathcal{ELC}$ under SFL with operators of the form* l-and, l-or, l-implies, *and* g-implies.

Below, we illustrated some cases for which a correct behaviour of the fuzzyDL reasoner is supposed so far.

Remark 2 *fuzzyDL provides correct results (modulo correctness of the underlying MILP solver) in the following cases:*

- *In SFL over $[0, 1]$ or L_n with KBs such that*
  - *there are no aggregation operators such as weighted sum, OWA operators (weighted sum, OWAs, or fuzzy integrals), and*
  - *there are no operators of the form* l-and, l-or, l-implies, *and* g-implies.
- *If the TBox is empty or lazy unfoldable.*

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