Syntax and Semantics of FuzzyDL

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1. Fuzzy operators. $\ominus, \oplus, \ominus$ and $\Rightarrow$ denote a t-norm, t-conorm, negation function and implication function respectively; $\alpha, \beta \in [0, 1]$.

Łukasiewicz negation	$\ominus_{Ł}\alpha$	$1 - \alpha$
Gödel t-norm	$\alpha \otimes_G \beta$	$\min\{\alpha, \beta\}$
Łukasiewicz t-norm	$\alpha \otimes_{Ł} \beta$	$\max\{\alpha + \beta - 1, 0\}$
Gödel t-conorm	$\alpha \oplus_G \beta$	$\max\{\alpha, \beta\}$
Łukasiewicz t-conorm	$\alpha \oplus_{Ł} \beta$	$\min\{\alpha + \beta, 1\}$
Gödel implication	$\alpha \Rightarrow_G \beta$	$\begin{cases} 1, & \text{if } \alpha \leq \beta \\ \beta, & \text{if } \alpha > \beta \end{cases}$
Łukasiewicz implication	$\alpha \Rightarrow_{Ł} \beta$	$\min\{1, 1 - \alpha + \beta\}$
Kleene-Dienes implication	$\alpha \Rightarrow_{KD} \beta$	$\max\{1 - \alpha, \beta\}$
Zadeh'set inclusion	$\alpha \Rightarrow_Z \beta$	1 iff $\alpha \leq \beta$, 0 otherwise

The reasoner can accept three different semantics, which are used to interpret $\ominus, \oplus, \ominus$ and $\Rightarrow$.

- Zadeh semantics: Łukasiewicz negation, Gödel t-norm, Gödel t-conorm and Kleene-Dienes implication (except in GCIs, where we have that the degree of membership to the subsumed concept should be less or equal than the degree of membership to the subsumer concept). This semantics is included for compatibility with earlier papers about fuzzy description logics.

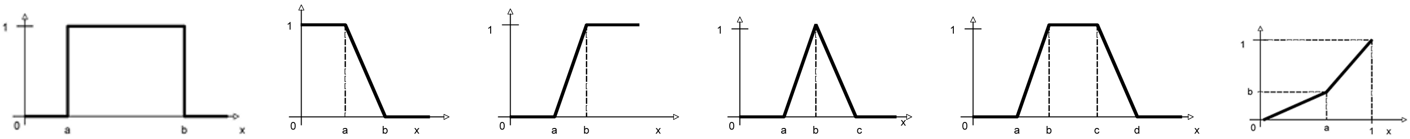
- Łukasiewicz semantics: Łukasiewicz negation, Łukasiewicz t-norm, Łukasiewicz t-conorm and Łukasiewicz implication.

- Classical semantics: classical (crisp) conjunction, disjunction, negation and implication.

Syntax to define the semantics of the knowledge base:

(fuzzy-logic [lukasiewicz — zadeh — classical])

2. Concrete Fuzzy Concepts. Concrete Fuzzy Concepts (CFCs) define a name for a fuzzy set with an explicit fuzzy membership function (we assume $a \leq b \leq c \leq d$).



(a) (b) (c) (d) (e) (f)

Figure 1: (a) Crisp value; (b) L-function; (c) R-function; (d) (b) Triangular function; (e) Trapezoidal function; (f) Linear hedge

(define-fuzzy-concept CFC crisp(k1,k2,a,b))	crisp interval (Figure 1 (a))
(define-fuzzy-concept CFC left-shoulder(k1,k2,a,b))	left-shoulder function (Figure 1 (b))
(define-fuzzy-concept CFC right-shoulder(k1,k2,a,b))	right-shoulder function (Figure 1 (c))
(define-fuzzy-number CFC triangular(k1,k2,a,b,c))	triangular function (Figure 1 (d))
(define-fuzzy-concept CFC trapezoidal(k1,k2,a,b,c,d))	trapezoidal function (Figure 1 (e))

3. Fuzzy Numbers. Firstly, if fuzzy numbers are used, one has to define the range $[k_1, k_2] \subseteq \mathbb{R}$ as follows:
`(define-fuzzy-number-range k1 k2)`
Let `fi` be a fuzzy number (a_i, b_i, c_i) $(a \leq b \leq c)$, and $n \in \mathbb{R}$. Valid fuzzy number expressions (see Figure 1 (d)) are:

name	fuzzy number definition	*name*
(a, b, c)	fuzzy number	(a, b, c)
n	real number	(n, n, n)
(f+ f1 f2 ... fn)	addition	$(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i)$
(f- f1 f2)	substraction	$(a_1 - c_2, b_1 - b_2, c_1 - a_2)$
(f* f1 f2 ... fn)	product	$(\prod_{i=1}^{n} a_i, \prod_{i=1}^{n} b_i, \prod_{i=1}^{n} c_i)$
(f/ f1 f2)	division	$(a_1/c_2, b_1/b_2, c_1/a_2)$

Fuzzy numbers can be named as:
`(define-fuzzy-number name fuzzyNumberExpression)`

4. Truth constants. Truth constants can be defined as follows (and later on, they can be used as the lower bound of a fuzzy axiom): (define-truth-constant constant n), where n is a rational number in $[0, 1]$.

5. Concept modifiers. Modifiers change the membership function of a fuzzy concept.

(define-modifier CM linear-modifier(b))	linear hedge with $b > 0$ (Figure 1 (f))
(define-modifier CM triangular-modifier(a,b,c))	triangular function (Figure 1 (d))

6. Features. Features are functional datatype attributes.

(functional F)	Firstly, the feature is defined. Then we set the range
(range F *integer* k1 k2)	The range is an integer number in $[k_1, k_2]$
(range F *real* k1 k2)	The range is a rational number in $[k_1, k_2]$
(range F *string*)	The range is a string

7. Datatype restrictions.

(>= var F)	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq var)]$
(>= F $f(F_1, \ldots, F_n)$)	at least datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(>= FN F)	at least datatype restriction	$\sup_{b, b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \geq b') \otimes FN^{\mathcal{I}}(b')$
(<= var F)	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq var)]$
(<= F $f(F_1, \ldots, F_n)$)	at most datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(<= FN F)	at most datatype restriction	$\sup_{b, b' \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b \leq b') \otimes FN^{\mathcal{I}}(b')$
(= var F)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b = var)]$
(= F $f(F_1, \ldots, F_n)$)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes (b = f(F_1, \ldots, F_n)^{\mathcal{I}})]$
(= FN F)	exact datatype restriction	$\sup_{b \in \Delta_{\mathbf{D}}} [F^{\mathcal{I}}(x, b) \otimes FN^{\mathcal{I}}(b)$

In datatype restrictions, the variable *var* may be replaced with a value (an integer, a real, or a string, depending on the range of the feature F).

Furthermore, is defined as follows:

$$
\begin{array}{rl}
f(F_1, \ldots, F_n) \rightarrow & F \\
& real \\
& (nF) \mid (n * F) \\
& (F_1 - F_2) \\
& (F_1 + F_2 + \ldots + F_n)
\end{array}
$$

8. Concept expressions.

top	top concept	1
bottom*	bottom concept	0
A	atomic concept	$A^{\mathcal{I}}(x)$
(and C1 C2)	concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes {C_2}^{\mathcal{I}}(x)$
(g-and C1 C2)	Gödel concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes_G {C_2}^{\mathcal{I}}(x)$
(l-and C1 C2)	Łukasiewicz concept conjunction	${C_1}^{\mathcal{I}}(x) \otimes_{Ł} {C_2}^{\mathcal{I}}(x)$
(or C1 C2)	concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus {C_2}^{\mathcal{I}}(x)$
(g-or C1 C2)	Gödel concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus_G {C_2}^{\mathcal{I}}(x)$
(l-or C1 C2)	Łukasiewicz concept disjunction	${C_1}^{\mathcal{I}}(x) \oplus_{Ł} {C_2}^{\mathcal{I}}(x)$
(not C1)	concept negation	$\ominus_{Ł} {C_1}^{\mathcal{I}}(x)$
(implies C1 C2)	concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x)$
(g-implies C1 C2)	Gödel concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x)$
(l-implies C1 C2)	Łukasiewicz concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x)$
(kd-implies C1 C2)	Kleene-Dienes concept implication	${C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x)$
(all R C1)	universal role restriction	$\inf_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, y) \Rightarrow {C_1}^{\mathcal{I}}(y)$
(some R C1)	existential role restriction	$\sup_{y \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(x, y) \otimes {C_1}^{\mathcal{I}}(y)$
(ua s C1)	upper approximation	$\sup_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)$
(la s C1)	lower approximation	$\inf_{y \in \Delta^{\mathcal{I}}} s^{\mathcal{I}}(x, y) \Rightarrow C^{\mathcal{I}}(y)$
(tua s C1)	tight upper approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x, z) \Rightarrow \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y, z) \otimes C^{\mathcal{I}}(y)\}\}$
(lua s C1)	loose upper approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x, z) \otimes \sup_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y, z) \otimes C^{\mathcal{I}}(y)\}\}$
(tla s C1)	tight lower approximation	$\inf_{z \in X} \{s_i^{\mathcal{I}}(x, z) \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y, z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(lla s C1)	loose lower approximation	$\sup_{z \in X} \{s_i^{\mathcal{I}}(x, z) \otimes \inf_{y \in \Delta^{\mathcal{I}}} \{s_i^{\mathcal{I}}(y, z) \Rightarrow C^{\mathcal{I}}(y)\}\}$
(self S)	local reflexivity concept	$S^{\mathcal{I}}(x)(x, x)$
(CM C1)	modifier applied to concept	$f_{\mathtt{m}}({C_1}^{\mathcal{I}}(x))$
(CFC)	concrete fuzzy concept	$CFC^{\mathcal{I}}(x)$
(FN)	fuzzy number	$FN^{\mathcal{I}}(x)$
(w-sum (n1 C1) ... (nk Ck))	weighted sum	$n_1 {C_1}^{\mathcal{I}}(x) + \ldots + n_k {C_k}^{\mathcal{I}}(x)$
(n C1)	weighted concept	$n {C_1}^{\mathcal{I}}(x)$
([>= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \geq w \\ 0, & \text{otherwise} \end{cases}$
([<= var] C1)	threshold concept	$\begin{cases} {C_1}^{\mathcal{I}}(x), & \text{if } {C_1}^{\mathcal{I}}(x) \leq w \\ 0, & \text{otherwise} \end{cases}$
(DR)	datatype restriction	$DR^{\mathcal{I}}(x)$

where $n_1, \ldots, n_k \in [0, 1]$ with $\sum_{i=1}^{k} n_i \leq 1$, and w is a variable or a real number in $[0, 1]$.

Fuzzy numbers can only appear in existential, universal and datatype restrictions.

Similarly, in threshold concepts *var* may be replaced with w.

Fuzzy relations s should be previously defined as fuzzy similarity relation or a fuzzy equivalence relation as `(define-fuzzy-similarity s)` or `(define-fuzzy-equivalence s)`, respectively.

Important note: The reasoner restricts the calculus to witnessed models.

9. Axioms.

(instance a C1 [d])	concept assertion	${C_1}^{\mathcal{I}}(a^{\mathcal{I}}) \geq d$
(related a b R [d])	role assertion	$R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq d$
(implies C1 C2 [d])	GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow {C_2}^{\mathcal{I}}(x) \geq d$
(g-implies C1 C2 [d])	Gödel GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_G {C_2}^{\mathcal{I}}(x) \geq d$
(kd-implies C1 C2 [d])	Kleene-Dienes GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{KD} {C_2}^{\mathcal{I}}(x) \geq d$
(l-implies C1 C2 [d])	Łukasiewicz GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_{Ł} {C_2}^{\mathcal{I}}(x) \geq d$
(z-implies C1 C2 [d])	Zadeh's set inclusion GCI	$\inf_{x \in \Delta^{\mathcal{I}}} {C_1}^{\mathcal{I}}(x) \Rightarrow_Z {C_2}^{\mathcal{I}}(x) \geq d$
(define-concept C1 C2)	concept definition	$\inf_{x \in \Delta^{\mathcal{I}}} {C_2}^{\mathcal{I}}(x) = {C_1}^{\mathcal{I}}(x)$
(define-primitive-concept C1 C2)	concept subsumption	$\inf_{x \in \Delta^{\mathcal{I}}} {C_2}^{\mathcal{I}}(x) \leq {C_1}^{\mathcal{I}}(x)$
(disjoint C1 ... Ck)	concept disjointness	$\forall_{i,j \in \{1,\ldots,k\}, i<j}$ (implies (g-and Ci Cj) *bottom*)
(disjoint-union C1 ... Ck)	disjoint union	(disjoint C2 ... Ck) and C1 = (or C2 ... Ck)
(range R C1)	range restriction	(implies *top* (all RN C))
(domain R C1)	fomain restriction	(implies (some RN *top*) C)
(functional R)	functional role	$R^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(a, c) \rightarrow b = c$
(reflexive R)	reflexive role	$\forall a \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, a) = 1.$
(symmetric R)	symmetric role	$\forall a, b \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(a, b) = R^{\mathcal{I}}(b, a).$
(transitive R)	transitive role	$\forall_{a,b \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a, b) \geq \sup_{c \in \Delta^{\mathcal{I}}} R^{\mathcal{I}}(a, c) \otimes R^{\mathcal{I}}(c, b).$
(implies-role R1 R2 [d])	RIA	$\inf_{x,y \in \Delta^{\mathcal{I}}} {R_1}^{\mathcal{I}}(x, y) \Rightarrow_{Ł} {R_2}^{\mathcal{I}}(x, y) \geq d$
(inverse R1 R2)	inverse role	${R_1}^{\mathcal{I}} \equiv ({R_2}^{\mathcal{I}})^{-}$

where d is the degree and can be: *(i)* a variable, *(ii)* an already defined truth constant, *(iii)* a rational number in $[0, 1]$, *(iv)* a linear expression.

Under Zadeh logic, $\Rightarrow$ is assumed to be Zadeh's set inclusion.
Important note: Transitive roles cannot be functional.

10. Queries.

(max-instance? a C)	$\sup\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(min-instance? a C)	$\inf\{n \mid \mathcal{K} \models$ (instance a C n)$\}$
(max-related? a b R)	$\sup\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(min-related? a b R)	$\inf\{n \mid \mathcal{K} \models$ (related a b R n)$\}$
(max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (implies D C n)$\}$
(g-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(g-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (g-implies D C n)$\}$
(l-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(l-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (l-implies D C n)$\}$
(kd-max-subs? C D)	$\sup\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(kd-min-subs? C D)	$\inf\{n \mid \mathcal{K} \models$ (kd-implies D C n)$\}$
(max-sat? C [a])	$\sup_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(min-sat? C [a])	$\inf_{\mathcal{I}} \sup_{a \in \Delta^{\mathcal{I}}} C^{\mathcal{I}}(a)$
(max-var? var)	$\sup\{$var $\mid \mathcal{K}$ is consistent$\}$
(min-var? var)	$\inf\{$var $\mid \mathcal{K}$ is consistent$\}$
(defuzzify-lom? C_m a F)	Defuzzify the value of F using largest of the maxima
(defuzzify-mom? C_m a F)	Defuzzify the value of F using middle of the maxima
(defuzzify-som? C_m a F)	Defuzzify the value of F using smallest of the maxima
(bnp? f)	Computes the Best Non-Fuzzy Performance (BNP) of fuzzy number f

where concept C_m represents several Mamdani/Rules IF-THEN fuzzy rules expressing how to obtain the value of concrete feature F.

11. Constraints. Constraints are of the form (constraints ⟨ constraint-i ⟩ +), where ⟨ constraint-i ⟩ is one of the following (with $OP = \geq \mid \leq \mid =$.):

(a1 * var1 + ... + ak * vark OP number)	linear inequation	$a_1 var_1 + \ldots + a_k * var_k \ OP \ number$
(binary var)	binary variable	$var \in \{0, 1\}$
(free var)	binary variable	$var \in (-\infty, \infty)$

12. Show statements.

(show-concrete-fillers F1 ... Fn)	show the value of the fillers of $F_1 \ldots F_n$
(show-concrete-fillers-for a F1 ... Fn)	show the value of the fillers of $F_1 \ldots F_n$ for a
(show-concrete-instance-for a F C1 ... Cn)	show the degrees of being the F filler of a an instance of C_i
(show-abstract-fillers R A1 ... An)	show the membership to $A_1, \ldots A_n$ of the fillers of R
(show-abstract-fillers-for a R A1 ... An)	show the membership to $A_1, \ldots A_n$ of the fillers of R for a
(show-variables x1 ... xn)	show the value of the variables $x_1 \ldots x_n$
(show-instances A1 ... An)	show the value of the instances of the atomic concepts $A_1 \ldots A_n$
(show-concepts a1 ... an)	show the membership of $a_1 \ldots a_n$ to any atomic concept
(show-language)	show the language of the KB, from $\mathcal{ALC}$ to $\mathcal{SHIF}(D)$

where C_i is the name of a defined concrete fuzzy concept. We assume that an abstract role R appears in at most one statement of the forms show-abstract-fillers? or show-abstract-fillers-for?.

13. Importing OWL Lite ontologies There is also the possibility to import OWL Lite ontologies, use `java OwlLoader <OwlLiteOntology> <outputFileName>`, with `<OwlLiteOntology>` being the URL of the OWL Lite ontology and `<fileName>` the name of the output file in fuzzyDL syntax.

14. Comments Any line beginning with # or % is considered a comment.